The International Conference on Computing Innovation and Applied Physics (CONF-CIAP 2022)
DOI: 10.54254/2753-8818/2/20220090

Some Fundamentals of Complex Analysis

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Abstract. In this paper, we first introduced the significance of complex variables as a field in the first section. Then, for the main body section, we explored the history and development of complex numbers. In addition, we demonstrated the key concepts and theorems that are crucial in the fundamentals of this field. We dive in depth to the such fundamental knowledge, which is the preliminaries of complex numbers. We analyzed the preliminaries via three smaller sections: complex numbers and the complex plane, functions on the complex plane, and integration along curves. Finally, we applied these knowledges into deriving a proof for the product of different sine expressions, which results in a generalized formula. Ultimately, the last section will be the conclusion of the paper.

Keywords: Complex Analysis, Complex Variables

1. Introduction

Complex Variables is a field of mathematics that deals with complex variables which include complex numbers, which are in the form a + bi, where a is the real part, b is the imaginary part, and a and b are both real numbers and $i^2=-1$. Also, this field covers various topics, from Cauchy’s theorem to prime number theorem [1-13]. It does not only help mathematicians, but it also assists physicists, engineers, and others in different fields to solve an enormous number of problems.

Before the invention of Complex Variables, mathematicians were wondering how to solve the quadratic equations such as $x^2 = -1$. Later, after the discoveries of the subject, mathematicians can solve all the quadratic equations. For example, if we encounter an equation $x^2-2x+10=0$, there is definitely no real solutions since the discriminant is negative. However, fortunately, we have complex numbers, helping us to find the roots of this equation successfully, which are $x=1\pm 3i$. Besides the quadratic equations, complex numbers have been an extremely helpful tool for other problems in mathematics. For example, what is the ln (-1) or, in Complex Variables, the log (-1)? With only the real numbers, there is definitely again no solution for this bewildered equation. Nevertheless, the complex numbers can help us to get the fascinating answer $i\pi(2n+1)$ for n belongs to integers. Also, Complex Variables can oftentimes apply to the real numbers, such as computing real integrals by applying the Residue Theorem and computing particular real-valued Improper Integrals by using the method of complex-valued functions [14]. What is more, Complex Variables also contributes to the fundamental theorem of algebra and Riemann hypothesis. Moreover, there is a magnificent identity called the Euler’s Identity,

DOI: 10.54254/2753-8818/2/20220090

which is $e^{i\pi} + 1 = 0$. This is beautiful since it combines e and π, which are irrational numbers, and i, an imaginary number, together to obtain a rational number, -1. Therefore, the field of Complex Variables offers math students, enthusiast, experts, and professionals with joys of solving problems that could not be solved by real numbers.

In physics, complex numbers appear in many topics, such as classical mechanics, special relativity, electromagnetism, and, especially, quantum mechanics. For examples, the complex numbers are included in the quantum mechanics' mathematical formulation [14]. In fluid dynamics, Complex Variables are applied to describe the potential flow in two dimensions fractals [14]. In engineering, Complex Variables mainly contributes to the Control Theory, Signal Analysis, structures in building, and stresses and strains on rays [14]. Furthermore, Complex Variables can help to solve differential equations that are vital to the fields of electrostatics, fluid dynamics, heat conduction, and etc. [14]. As a result, there is a huge amount scientist in various areas of science using the subject of Complex Variables to assist their studies.

In conclusion, a fact that is highly interesting about the Complex Variables is that it is not directly relevant to the other contents. In other words, all subjects that can be measured, such as the equations, the electric field and magnetic field, and the velocities and positions, are real and the complex numbers, which cannot be measured, are not, but they perfectly connect to many other contents or subjects. There are many other profound applications of Complex Variables that are not mentioned above, so we can conclude that the study of Complex Variables is extremely significant to the whole human society.

2. Discussion

2.1. History of Complex Numbers

The origin of complex numbers, simply put, is from polynomial equations, specifically quadratic and cubic equations. Dating back to as early as the Babylonians of the 2000 BC [15], the generalized quadratic equation

$$ax^2 + bx + c = 0 \tag{1}$$

can be already solved using the quadratic formula, that is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{2}$$

However, at that time, people only considered positive roots. In other words, they only considered actual intersections of quadratics with linear functions. The number of intersections determines the number of solutions. They simply ignore any possible outcomes that include negative numbers. This is because negative numbers were not introduced to the world until the 16th century [1-15].

Only to this century did the first appearance of complex numbers, or a square root with a negative number underneath emerge. This primarily emerged from the works of Scipione del Ferro, Niccolo Tartaglia and Gerolamo Cardano [1-15]. With combined effort, these Italian mathematicians derived a formula for solving cubic equations. One of the general equation or case they considered is

$$x^3 = ax + b \tag{3}$$

The formula used to solve this is expressed as

$$x = \sqrt[3]{\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} - \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}} \tag{4}$$

This formula can be used to solve any cubic expressions taking the form shown above.

However, the actual wonder came years later. An Italian engineer, Rafael Bombelli, noticed a puzzle inside the formula. First, he came up with an equation

DOI: 10.54254/2753-8818/2/20220090

$$x^3 = 15x + 4 \tag{5}$$

This satisfies the form stated above, with $a = 15$ and $b = 4$. Then, he tried using the formula derived by Cardano, and came across the result

$$x = \sqrt[3]{2 + \sqrt{-121}} - \sqrt[3]{2 - \sqrt{-121}} \tag{6}$$

This is the first time he encountered a negative number within a square root. At that time, people had not considered this mysterious expression, nor do they know any ways that could help solve this. Without perfect understanding, Bombelli eventually solved the equation. He did this by viewing $\sqrt{-121}$, or after simplification, $\sqrt{-1}$ as a real number [1-15]. He applied nothing more than the traditional mathematical techniques into this scenario. Afterwards, he derived the equality that

$$\sqrt[3]{2 + \sqrt{-121}} = 2 + \sqrt{-1} \tag{7}$$

and

$$\sqrt[3]{2 - \sqrt{-121}} = 2 - \sqrt{-1} \tag{8}$$

Now, he successfully solved the cubic equation, and determined that

$$x = 2 + \sqrt{-1} + 2 - \sqrt{-1} = 4 \tag{9}$$

Here, his method was to cancel out the positive and negative $\sqrt{-1}$ in the expression. Although this did not suffice to gain a complete understanding of complex and imaginary numbers, but it opened the road for further mathematical pursuit. Treating square roots of negative numbers became more pervasive if people were to solve all kinds of polynomial equations. Bombelli showed that, when using certain techniques, the complex numbers can be solved [1-15].

In the later centuries, complex numbers began to gain status in the field of mathematics. A series of mathematicians made pioneering contributions to complex numbers.

During the 17th century, French philosopher Rene Descartes innovated a new mathematical term in the study of complex numbers, called “imaginary”. He knew that the number of roots to a polynomial equation is equivalent to the degree of the polynomial. This does not form consistency with the intersections of functions through geometric analysis. Therefore, he concluded there are missing roots which we could only imagine [15].

As time progresses, English mathematician John Wallis used a geometric approach to demonstrate complex numbers, resulting in the formation of the complex plane, where the horizontal axis represents the real part of complex numbers and the vertical axis is the imaginary part. He assigned complex numbers two dimensions and visualized them as vectors on such a plane [15].

In the 18th century, Leonhard Euler make use of the complex plane and represented complex numbers as

$$z = x + iy \tag{10}$$

and

$$re^{i\theta} = r(cos\,\theta + i\,sin\,\theta) \tag{11}$$

In the 19th century, Carl Friedrich Gauss presented the new mathematical term “complex numbers” and William Rowan Hamilton used an algebraic approach to compute sums and product of complex numbers [15]. He used coordinate representation to demonstrate that

$$(a, b) + (c, d) = (a + c, b + d) \tag{12}$$

and

$$(a, b)(c, d) = (ac - bd, bc + ad) \tag{13}$$

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Being one of the last people who made a breakthrough in complex numbers, Augustin-Louis Cauchy combined the study of complex numbers with the study of integral calculus [15]. With the help of complex numbers, previously inexplicable integral can now be solved, including

$$\int_0^\infty \frac{\sin\theta}{\theta} d\theta = \frac{\pi}{2} \tag{14}$$

and

$$\int_0^\pi \log(\sin\theta)\, d\theta = -\pi \log 2 \tag{15}$$

2.2. Key Concepts of Complex Numbers

After an introduction to basic algebraic properties of complex numbers including moduli, conjugates, exponential form, argument, roots and geometric interpretation of complex numbers, calculus and integral of various elementary functions of a complex variable are studied to help solve equations of complex variables. This starts by defining a complex exponential function and then uses it to develop logarithm function, power function and trigonometric function. During this, some basic properties and theorems of logarithms and trigonometric functions are explored, and applied to deal with the domain of complex functions.

To make it simple to deal with a function $f(z)$ that the value of the integral of it around a simple closed contour is zero, the Cauchy-Goursat theorem is introduced:

If a function $f(z)$ is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\, dz = 0 \tag{16}$$

Then, another fundamental result will now be established -- the Cauchy integral formula.

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0} \tag{17}$$

Taylor's theorem states that suppose that a function $f(z)$ is analytic throughout a disk $|z - z_0| < R$, centered at z_0 and with radius R_0, then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_0 (z - z_0)^n \quad (|z - z_0| < R_0) \tag{18}$$

where

$$a_0 = \frac{f^{(n)}(z_0)}{n!} \quad (n = 0,1,2,\dots)$$

With the agreement that $f^{(0)}(z_0) = f(z_0)$, and $0! = 1$, and series can be written

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots \quad (|z - z_0| < R_0) \tag{19}$$

Laurent's theorem is therefore introduced to enable us to expand a function $f(z)$ into a series involving positive and negative powers of $(z - z_0)$ when the function fails to be analytic at z_0.

Suppose that a function $f(z)$ is analytic throughout an annular domain $R_1 < |z - z_0| < R_2$, centered at z_0, and let C denote any positively oriented simple closed contour around z_0 and lying in that domain. Then, at each point in the domain, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_0 (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_0}{(z-z_0)^n} \quad (R_1 < |z - z_0| < R_2) \tag{20}$$

where

$$a_0 = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \quad (n = 0,1,2,\dots)$$

and

DOI: 10.54254/2753-8818/2/20220090

$$b_0 = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}} \ (n = 1,2,...)$$

The Cauchy's residue theorem:

Let C be a simple closed contour, described in the positive sense. If a function $f(z)$ is analytic inside and on C except for a finite number of singular points a_k ($k = 1,2,...,n$) inside C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res(f, a_k) \tag{21}$$

Finally, we have come to the most interesting part - how to find the integral of f (z) dz over three different contours using Cauchy's residue theorem.

For example [2], let's see the problem first.

Define

$$f(z) = \frac{az^3+bz^2+cz+d}{z^4-1} \tag{22}$$

with

$$a = 10, b = -3 + i, c = 0, d = -3 - I$$

Evaluate the integral

$$\int_\delta f(z)dz$$

where

$$\delta(t) = 1 + e^{it}, 0 \ll t \ll 2\pi,$$

$$\delta(t) = \frac{1-i}{2} + \sqrt{2}e^{it}, 0 \ll t \ll 2\pi,$$

$$\delta(t) = 2e^{it}, 0 \ll t \ll 2\pi,$$

To find the integral of f(z) around the contour, we need to know the singularities of f(z) first. Since the numerator is an entire function, this function has singularities only where the denominator $z^4 - 1$ is zero.

Since $z^4 - 1 = (z-1)(z-i)(z+1)(z+i)$, that happens only where $z = 1, z = i, z = -1, z = -i$

We can rewrite the function in a partial-function way as follows:

$$f(z) = \frac{A}{z-1} + \frac{B}{z-i} + \frac{C}{z+1} + \frac{D}{z+i} \tag{23}$$

Then we need to find the exact value of A, B, C, and D.

To reach this, we reduce the four fractions to a common denominator.

$$f(z) = \frac{A(z-i)(z+1)(z+i)+B(z-1)(z+1)(z+i)+C(z-1)(z-i)(z+i)+D(z-1)(z-i)(z+1)}{(z-1)(z-i)(z+1)(z+i)} \tag{24}$$

And expand the numerator to the same form of the original function to make comparisons.

$$f(z) = \frac{z^3(A+B+C+D)+z^2(A+iB-C-iD)+z(A-B+C-D)+(A-iB-C+iD)}{(z-1)(z-i)(z+1)(z+i)} \tag{25}$$

Then we can get the following equations.

$$\begin{aligned} a &= A + B + C + D \\ b &= A + iB - C - Id \\ c &= A - B + C - D \\ d &= A - iB - C + iD \end{aligned} \tag{26}$$

However, it is very difficult to solve four equations with four variables. So, in an easier way, we can apply this in a matrix form [3].

$$M\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=\begin{pmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{pmatrix}\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=\begin{pmatrix}a\\b\\c\\d\end{pmatrix} \tag{27}$$

We needed to multiply both sides of the equality by the inverse of the matrix. In this case, we replace all the complex numbers of the original matrix by their conjugates and multiply by a quarter.

$$M^{-1}=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{pmatrix} \tag{28}$$

Now we get

$$\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=M^{-1}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\frac{1}{4}\begin{pmatrix}1&1&1&1\\1&i&-1&-i\\1&-1&1&-1\\1&-i&-1&i\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

Now we organize the previous matrix into equation form by substituting the value of a, b, c, d respectively to calculate the value of A, B, C, D.

After matrix computation, we end up with these expressions.

$$A=\frac{a+b+c+d}{4}=\frac{10-3+i-3-i}{4}=\frac{4}{4}=1 \tag{29}$$
$$B=\frac{a-ib-c+id}{4}=\frac{10+3i+1-3i+1}{4}=\frac{12}{4}=3$$
$$C=\frac{a-b+c-d}{4}=\frac{10+3-i+3+i}{4}=\frac{16}{4}=4$$
$$D=\frac{a+ib-c-id}{4}=\frac{10-3i-1+3i-1}{4}=\frac{8}{4}=2$$

As a result, we can now define $f(z)$ in terms of these four expressions.

$$f(z)=\frac{1}{z-1}+\frac{3}{z-i}+\frac{4}{z+1}+\frac{2}{z+i} \tag{30}$$

Then, we use the Cauchy's residue theorem to solve the integrals in three different contours.

For $\delta(t)=1+e^{it}, 0\ll t\ll 2\pi$, in a complex plane, this is geometrically a circle with radius 1 centered at $(1,0)$. From the diagram, we can observe that $(1,0)$is contained in the contour. So, the residue of this point is simply A. The resulting answer is:

$$\int f(z)dz = 2\pi i\,(Res(f,1)) = 2\pi iA = 2\pi i \tag{31}$$

For $\delta(t)=\frac{1-i}{2}+\sqrt{2}e^{it}, 0\ll t\ll 2\pi$, in a complex plane, this is a circle with center at $\left(\frac{1}{2},-\frac{1}{2}\right)$, having a radius of $\sqrt{2}$. It encloses the non-differentiable points $z=1, z=-i$. So, the residue of these points is simply A+D. The resulting answer is:

$$\int f(z)dz = 2\pi i\,(Res(f,1)+Res(f,-i)) = 2\pi i(A+D)= 6\pi i \tag{32}$$

For $\delta(t)=2e^{it}, 0\ll t\ll 2\pi$, in a complex plane, this is a circle with center at the origin with a radius of 2. From the diagram, we can observe that all of the non-differentiable points $z=1, z=i, z=$

DOI: 10.54254/2753-8818/2/20220090

$-1, z = -i$ are contained in the contour. So, the residue of these points is simply A+B+C+D, while we know that $a = A + B + C + D = 10$. The resulting answer is:

$$\int f(z)dz = 2\pi i\,(Res(f,1) + Res(f,-1) + Res(f,-i) + Res(f,i) = 2\pi i(A + B + C + D) = 20\pi i \quad (33)$$

2.3. Detailed Preliminaries of Complex Numbers

2.3.1. Complex Numbers and the Complex Plane

By definition, complex numbers are the set of numbers denoted as

$$C \coloneqq \{x + iy: x, y \in R\} \quad (34)$$

where i is defined as $\sqrt{-1}$. The variable x in complex numbers are the real part of complex numbers, represented as $Re(z)$. On the other hand, the variable y in them are the imaginary part, expressed as $Im(z)$. If $Re(z) = 0$, then the complex number is imaginary. If $Im(z) = 0$, then it is real. The absolute value of complex numbers shows its magnitude, such that

$$|z| \coloneqq \sqrt{x^2 + y^2} \quad (35)$$

Complex numbers can also be applied with operations. For addition, sum of real parts of complex numbers constitute the real part of the resulting complex number, while sum of imaginary parts of them forms the resulting imaginary part, such that

$$(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2) \quad (36)$$

Multiplication works in the same as taking the product of two linear expressions, which is

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \quad (37)$$

Moreover, many identities are developed for complex numbers with the introduction of the complex conjugate, which is

$$\overline{x + \imath y} = x - iy \quad (38)$$

The list of identities can be shown as

$$\overline{z + w} = \bar{z} + \bar{w} \quad (39)$$

$$\overline{zw} = \bar{z}\bar{w}$$

$$Re(z) = \frac{1}{2}(z + \bar{z})$$

$$Im(z) = \frac{1}{2i}(z - \bar{z})$$

$$|z| = \sqrt{z\bar{z}}$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

$$|zw| = |z||w|$$

Complex numbers can be visualized using a coordinate plane, with horizontal axis being the real part and the vertical axis being the imaginary part. For example, the complex expression $x + iy$ has coordinate (x, y) on a complex plane.

With such visualization, we can easily see how addition of complex numbers can be operated. Complex numbers in a complex plane act as vectors pointing from the origin to the coordinate (x, y). The addition of them is exactly the same as vector additions.

However, in order to visualize multiplication, we need to introduce an alternative form of complex numbers, the polar form. By definition, the polar form of complex number can be written as

$$z = |z|e^{i\theta}, z \in C, \theta \in R \tag{40}$$

Here, the angle θ is called the argument of z, which is shown as $\arg(z)$.

With basic exponential and trigonometric identities, we can see that the product of $e^{i\theta_1}$ and $e^{i\theta_2}$ results in the expression $e^{i(\theta_1+\theta_2)}$. Now, we can perform multiplication operation in a simpler way, such that

$$zw = |z|e^{i\,arg(z)}|w|e^{i\,arg(w)} = |zw|e^{i(arg(z)+arg(w))} \tag{41}$$

Furthermore, we can also experiment inequalities with complex numbers. The most important inequality is the triangle inequality. In a triangle, we cannot have the fact that the sum of two side lengths is greater or equal to the that the third side. Similarly, when taking the modulus of complex numbers after addition, we must get

$$|z + w| \leq |z| + |w| \tag{42}$$

After manipulating with different inequalities obtained, we can get the final result

$$||z| - |w|| \leq |z - w| \tag{43}$$

Complex numbers, just like functions encountered in calculus, have convergence. By definition, a complex number sequence z_n converges to the complex number ω when

$$\lim_{n\to\infty} |z_n - \omega| = 0 \tag{44}$$

or, in other words,

$$\lim_{n\to\infty} z_n = \omega$$

Alternatively, due to the previous discussion of inequalities, we may also write

$$\lim_{n\to\infty} Re(z_n) = \omega \tag{45}$$

$$\lim_{n\to\infty} Im(z_n) = \omega$$

We can also develop the concept of Cauchy's sequence, which further explores convergence. By definition, $z_n \subset \mathrm{C}$, where $n \in N$, is a Cauchy sequence if for all $\varepsilon > 0$ and there exists a real number N' with the fact that for all $n, m \geq N'$ we can get

$$|z_n - z_m| < \varepsilon \tag{46}$$

This means that as we progress further with the terms of a Cauchy sequence, we can see that they become more and more closer together and close to the limit, resulting in a convergent sequence.

Complex numbers also have a variety of interpretations in element sets and topology. By definition, suppose we have $z_0 \in C, r > 0$, the open disc, closed disc and circle with radius r and center z_0 are respectively

$$D_r(z_0) \coloneqq \{z \in C: |z - z_0| < r\} \tag{47}$$

$$\overline{D}_r(z_0) \coloneqq \{z \in C: |z - z_0| \leq r\}$$

$$C_r(z_0) \coloneqq \{z \in C: |z - z_0| = r\}$$

With the above definitions, we can derive the unit disc

$$D_1(0) \coloneqq \{z \in C: |z| < 1\} \tag{48}$$

From these, we can build additional definitions for open and closed subsets. For the subsets

$$\Omega \subset C \tag{49}$$

DOI: 10.54254/2753-8818/2/20220090

if the radius $r > 0$ for $z \in \Omega$ and we can derive

$$D_r(z) \subset \Omega \tag{50}$$

then the subset is called open. On the other hand, if the complement of the subset $\Omega^c = C\backslash\Omega$ is open, then the subset itself is called closed.

Furthermore, we also have the definitions for additional key concepts.

For interior point, we can consider $z \in C$ to be one if $r > 0$ and $D_r(z) \subset \Omega$. The set of all interior points is called the interior, denoted as Ω^o.

For limit, z is to be considered if there exists a sequence $z_n \subset \Omega\backslash\{z\}$ and $\lim_{n\to\infty} z_n = z$. The closure, $\overline{\Omega}$, is the union of Ω and all the limit points present.

The boundary takes into account the interior and closure, such that

$$\partial\Omega := \bar{\Omega}\backslash\Omega^o \tag{51}$$

On top of that, one of the key concepts in complex numbers is path, this relates to analyzing functions on a complex plane. In addition, it offers a precise definition of connectedness and region in such a plane. For $z, w \in \Omega$, a path going from z to w is a continuous function $f: [a, b] \to \Omega$, where $f(a) = z$ and $f(b) = w$ for the interval $[a, b] \in R$. Also, if any combination of points z, w can form a path, then the subset Ω is described as path connected.

2.3.2. Functions on the Complex Plane

Now, here is an overall review of the functions of complex variables.

So, what is a function, which can be also referred to a relation in this case, of a complex variable? It is simply mapping the complex number *z*, *x+yi*, to another complex number *w*, *u+bi* or *u(x,y)+v(x,y)i*. For example, let us consider

$$F(z) = z^2, where\ z = x + yi \tag{52}$$

$$f(z)=(x+yi)^2$$

$$=x^2+2xyi+(yi)^2$$

$$=x^2-y^2+2xyi$$

After we had obtained this answer, we map x^2-y^2 to *u* and *2xy* to *v*.

$$f(z)=u(x,y)+v(x,y)i, \text{ where } u=x^2-y^2 \text{ and } v=2xy \tag{53}$$

This is simply organizing the *f(z)*, indicating that x^2-y^2 is the real part and *2xy* is the imaginary part, so that we can visualize it better on the complex plane.

Since we already have some ideas about a function of a complex variable, we should start with the derivative of a complex function *f'(z)*.

The definition of *f'(z)*:

$$f'(z)=\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} \tag{54}$$

Pretty much the same as when we are dealing with the derivative of a function *f(x)* in Calculus. However, there is a difference between *f'(z)* and *f'(x)*. When *f(x)* is differentiable, it includes that the horizontal limits which approach a point of *f(z)* should have the same value since *x* exists on a number line. On the contrary, when *f(z)* is differentiable, it states that the limits from all directions which approach a point of *f(z)* should have the same value because a complex number does not merely exist on a number line, instead it exists on a whole plane. Moreover, when a contour *C* of *f(z)* in the complex plane contains a unique derivative at every point in *C*, we say that *f(z)* is holomorphic in *C*.

Now, let us get more familiar with the complex differentiability. Is $f(z)=z^2$ is holomorphic?

DOI: 10.54254/2753-8818/2/20220090

$$f'(z)=\lim_{h\to 0}\frac{(z+h)^3-z^3}{h} \tag{55}$$

$$=\lim_{h\to 0}\frac{z^3+3z^2h+3zh^2+h^3-z^3}{h}$$

$$=\lim_{h\to 0}3z^2+3zh+h^2 = 3z^2$$

Therefore, $f(z)$ in this case is holomorphic. In fact, in general, $f(z)=z^n$ is holomorphic and thus $f'(z)=nz^{n-1}$, where $n\in Z$. Next, is $f(z)=2y+xi$ holomorphic?

Note that here $2y=u(x,y)$ *and* $x=v(x,y)$. *Hence,* $f(z+\Delta z)=2(y+\Delta y)+(x+\Delta x)i$, *meaning that* $f(z+\Delta z)-f(z)=2\Delta y+(\Delta x)i$ *and thus we get the following:*

$$f'(z)=\lim_{h\to 0}\frac{2\Delta y+(\Delta x)i}{\Delta x+(\Delta y)i} \text{ since } \Delta z=\Delta x+(\Delta y)i \tag{56}$$

Now we deal with the situation where $\Delta z\to 0$ along Re(z) and thus $\Delta y=0$:

$$f'(z)=\left[\frac{(\Delta x)i}{\Delta x}\right]_{\Delta x\to 0} = i \tag{57}$$

Next we deal with the situation where $\Delta z\to 0$ along Im(z) and thus $\Delta x=0$:

$$f'(z)=\left[\frac{2\Delta y}{(\Delta y)i}\right]_{\Delta y\to 0} = \frac{2}{i} = -2i \tag{58}$$

Since two situations' answers are different, we can make the conclusion that $f(z)$ in this case is not holomorphic.

Finally, we need a general equation for calculating the complex differentiability of a complex function so that in most cases we can save our time. The equation obtained from steps we did above, which include horizontal approach and vertical approach:

Cauchy-Riemann Equations [5]:

$$u_x(x,y) = v_y(x,y) \text{ or } u_x = v_y \tag{59}$$

$$u_y(x,y) = -v_x(x,y) \text{ or } u_y = -v_x$$

the subscript x or y refers to taking the first partial derivative respect to x or y.

What these equations mean is that if $f(z)=u(x,y)+iv(x,y)$ and it is holomorphic at a point where $z_0=x_0+iy_0$. Then the first-order partial derivatives of u and v must exist at (x_0,y_0), and they have to conform to the Cauchy-Riemann equations above [2].

2.3.3. Integration along Curves [4]

Proof of Cauchy-Goursat theorem [6]:

We first break the integral f into its real and imaginary components:

$$f(z) = u + iv \tag{60}$$

In this case,

$$\int_C f(z)\, dz = \int_C (u+iv)\,(dx+idy) = \int_C (udx - vdy) + i\int_C (vdx+udy) \tag{61}$$

We then need to use Green's Theorem:

$$\int_C (udx+vdy) = \iint_D (v_x - u_y)dx\,dy \tag{62}$$

We may then replace the integrals around the closed contour C with an area integral throughout the domain d that is enclosed by C as follows:

DOI: 10.54254/2753-8818/2/20220090

$$\int_C (udx - vdy) = \iint_D (-v_x - u_y)dx\,dy \tag{63}$$
$$\int_C (vdx + udy) = \iint_D (u_x - v_y)dx\,dy$$

As the function $f(z)$ is holomorphic in the domain D, u and v may satisfy the Cauchy–Riemann equations:

$$u_x = v_y \tag{64}$$
$$u_y = -v_x$$

By substituting these, we finally find that both integrals are zero.

$$\iint_D (-v_x - u_y)dx\,dy = \iint_D (u_y - u_y)dx\,dy = 0 \tag{65}$$
$$\iint_D (u_x - v_y)dx\,dy = \iint_D (u_x - u_x)dx\,dy = 0$$

This gives he desired result.

$$\int_C f(z)\,dz = 0 \tag{66}$$

Then, another fundamental result will now be established -- the Cauchy integral formula.

Proof of Cauchy integral formula [7]:

Let C_ρ denote a positively oriented circle $|z - z_0| = \rho$ and make sure ρ is small enough that C_ρ is interior to C. Note that quotient $\frac{f(z)}{z-z_0}$ is analytic between and on the contours C_ρ and C, then we have

$$\int_C \frac{f(z)dz}{z-z_0} = \int_{C_\rho} \frac{f(z)dz}{z-z_0} \tag{67}$$

Since we know

$$\int_{C_\rho} \frac{f(z)dz}{z-z_0} = 2\pi i$$

we can write

$$\int_C \frac{f(z)dz}{z-z_0} - f(z_0)\int_{C_\rho} \frac{dz}{z-z_0} = \int_C \frac{f(z)dz}{z-z_0} - 2\pi i f(z_0) = \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0} dz \tag{68}$$

Note that f is analytic, thus continuous, at z_0, we could have

$$|z_0 - f(z_0)| < \varepsilon \text{ whenever } |z - z_0| < \delta$$

By obtaining the upper bound for the moduli of contour integrals,

$$\left| \int_C \frac{f(z)dz}{z-z_0} - 2\pi i f(z_0) \right| = \left| \int_{C_\rho} \frac{f(z)-f(z_0)}{z-z_0} dz \right| < \frac{\varepsilon}{\rho} 2\pi\rho = 2\pi\varepsilon \tag{69}$$

we can know that the left-hand side of this inequality that is a nonnegative constant is less than an arbitrarily small positive number. Therefore, the only possible value of the left-hand side is zero.

$$\int_C \frac{f(z)dz}{z-z_0} - 2\pi i f(z_0) = 0 \tag{70}$$

By moving the subtracted part to the right-hand side, the Cauchy integral formula is thus obtained.

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-z_0} \tag{71}$$

Proof of Taylor's theorem:

In the case $z_0 = 0$, let $|z| = r$,

$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{f(s)ds}{s-z} \tag{72}$$

Now we need to process the factor $\frac{1}{s-z}$.

$$\frac{1}{s-z} = \frac{1}{s} \cdot \frac{1}{1-\frac{z}{s}} \tag{73}$$

Given that

$$\frac{1}{1-z} = \sum_{n=0}^{N-1} z^n + \frac{z^N}{1-z} \tag{74}$$

replacing z by $\frac{z}{s}$ and we can have

$$\frac{1}{s-z} = \sum_{n=0}^{N-1} \frac{1}{s^{n+1}} z^n + \frac{z^N}{(s-z)s^N} \tag{75}$$

Multiply by $f(s)$ and then integrate each side with respect to s around C_0.

$$\int_{C_0} \frac{f(s)ds}{s-z} = \sum_{n=0}^{N-1} \int_{C_0} \frac{f(s)ds}{s^{n+1}} z^n + z^N \int_{C_0} \frac{f(s)ds}{(s-z)s^N} \tag{76}$$

Note the fact that

$$\frac{1}{2\pi i} \int_{C_0} \frac{f(s)ds}{s^{n+1}} = \frac{f^{(n)}(0)}{n!} \quad (n = 0,1,2,\dots) \tag{77}$$

By multiplying through by $\frac{1}{2\pi i}$,

$$f(z) = \sum_{n=0}^{N-1} \frac{f^{(n)}(0)}{n!} + \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)ds}{(s-z)s^N} \tag{78}$$

In order to show that

$$\lim_{N\to\infty} \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)ds}{(s-z)s^N} = 0 \tag{79}$$

we see that $|s-z| \geq ||s|-|z|| = r_0 - r$.
Then, if P denotes the maximum value of $|f(s)|$ on C_0,

$$\left| \frac{z^N}{2\pi i} \int_{C_0} \frac{f(s)ds}{(s-z)s^N} \right| \leq \frac{r^N}{2\pi} \cdot \frac{P}{(r_0-r){r_0}^N} 2\pi r_0 = \frac{P r_0}{(r_0-r)} \left(\frac{r}{r_0}\right)^N \tag{80}$$

As $\frac{r}{r_0} < 1$, the limit holds.
Therefore, Taylor's theorem is derived.

Laurent's theorem [9] is therefore introduced to enable us to expand a function $f(z)$ into a series involving positive and negative powers of $(z-z_0)$ when the function fails to be analytic at z_0.

$$f(z) = \sum_{n=0}^{\infty} a_0 (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_0}{(z-z_0)^n} \quad (R_1 < |z-z_0| < R_2) \tag{81}$$

where

$$a_0 = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \ (n = 0,1,2,\dots) \tag{82}$$

and

$$b_0 = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{-n+1}} \ (n = 1,2,\dots) \tag{83}$$

Proof of Cauchy's residue theorem [10]:

Suppose that the points z_k $(k = 1,2,\dots,n)$ are centers of positively oriented circles C_k (interior to C and does not intersect with each other), we adapt the Cauchy-Goursat theorem within the domains.

$$\int_C f(z)\,dz - \sum_{k=1}^{n} \int_{C_k} f(z)\,dz = 0$$

This can reduce to

$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} Res(f, a_k) \tag{84}$$

since

$$\int_{C_k} f(z)\,dz = 2\pi i Res_{z=z_k} f(z) \ \ (k = 1,2,\ldots,n) \tag{85}$$

2.4. Proof of Product of Sine Expressions

Prove that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}} \tag{86}$$

Since

$$\sin z = \frac{e^{zi}-e^{-zi}}{2i} \tag{87}$$

We can have

$$\prod_{j=1}^{n-1}\sin\frac{j\pi}{n} = \prod_{j=1}^{n-1}\left(\frac{e^{\frac{ji\pi}{n}}-e^{-\frac{ji\pi}{n}}}{2i}\right) = \prod_{j=1}^{n-1}\left[\frac{e^{\frac{ji\pi}{n}}(e^{\frac{2ji\pi}{n}}-1)}{2i}\right] \tag{88}$$

Since $x^n - 1 = (x-1)(x^{n-1} + \cdots + x + 1)$, and the roots of $(x^{n-1} + \cdots + x + 1)$ are the nth roots of unity, we can derive from it that

$$(x^{n-1} + \cdots + x + 1) = \prod_{j=1}^{n-1} x - wi \tag{89}$$

where w is an nth roots of unity [11].

Let $x = 1$, the equation reduces to

$$\prod_{j=1}^{n-1} 1 - wi = n \tag{90}$$

Now we process

$$\begin{aligned}
&\left|1-e^{i\theta}\right| \\
&= \left|-e^{\frac{i\theta}{2}}\left(e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}\right)\right| \\
&= \left|-e^{\frac{i\theta}{2}}\right|\left|e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}\right| \\
&= |e^{\frac{i\theta}{2}} - e^{\frac{-i\theta}{2}}| \\
&= \left|2i\sin\frac{\theta}{2}\right| \\
&= 2\left|\sin\frac{\theta}{2}\right| \\
&= 2\sin\frac{\theta}{2}
\end{aligned}$$

since $\sin\frac{\theta}{2} > 0$ for $\frac{\theta}{2} \in [0, \pi]$

Let $w^k = e^{\frac{2ki\pi}{n}} = e^{i\theta}$, we can obtain that $\left|1 - w^k\right| = 2\sin\frac{k\pi}{n}$.

Therefore,

$$\prod_{j=1}^{n-1} 2\sin\frac{j\pi}{n} = \prod_{j=1}^{n-1}(1 - w^k) = n \tag{91}$$

This reduces to

DOI: 10.54254/2753-8818/2/20220090

$$\prod_{j=1}^{n-1} sin\frac{j\pi}{n} = \frac{n}{2^{n-1}} \quad (92)$$

3. Conclusion

In conclusion, the subject of Complex Variables is highly applicable and useful to various areas of study. In Complex Variables, mathematicians and scientists deal with complex numbers, which are not real-valued numbers and appear in the form a+bi. Besides complex numbers, there are complex functions, which all appear in the form f(z)=u(x,y)+v(x,y). Also, like real-valued functions, complex functions can be differentiated and integrated if some conditions are met.

In the future, we all would like to dedicate our times and energies into the study of mathematics, and some, in the meanwhile, would also like to dedicate in other areas that are deeply related to the study, such as Physics. In particular, as a mathematician, we will do researches about Complex Variables after we have acquired a more decent knowledge about this fascinating topic. As scientists, we will do researches about topics that are highly related to Complex Variables.

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