A Fundamental Tour of Complex-valued World

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Abstract. Throughout the learning, the topic of complex variables has always been a very special area in mathematics. As progressed through the complex variable courses, our group started to raise our interest in this area and wish to learn further. Hence, this essay will focus on several perspectives on the topic of complex variables. This will include the historical development, application, general theorem and solving real problems.

Keywords: Complex variables, Historical development and application

1. Background and application of complex variable

Complex numbers are found by an Italian mathematician called Gerolamo Cardanoin. We cannot measure complex number in real life, however, like negative numbers, it is really an important concept. Complex number is basically a combination of real and imaginary part, can be written as the form of "a+bi". In this number, "a" and "b" is real number, and "i" is an imaginary number, which also means the square root of "-1". complex number is very useful and important in many regions. By using complex numbers, we can solve some hard problems in a short period of time, and that is one of the reasons why scientists take complex numbers as an important tool nowadays. There are many properties according to complex number, and some of them have significant meaning. One of the properties is that complex numbers can be defined as a vector, means we can draw a graph that contain one real axis and one imaginary axis to show that number, and this graph is called Argand diagram. From this property, we can describe rotation problems by draw it on the diagram, which can make us solve those problems efficiently. The second properties of complex number can be used in Maclaurin series. Maclaurin series is a non-polynomial that are made by infinite many of polynomials. A famous mathematician called Euler use complex number to rewrite Maclaurin series into an exponential form, and the benefit of that new form of series is that it makes more mathematical operation possible. Furthermore, there are many applications of complex numbers in variety subjects related to science. A word in Electrical Engineering called "phase" is defined as the distance shift of sinusoid. By using complex numbers, functions that are different in phases can be calculated easily using simple methods, and it can help us understand the meaning of circuit. In Signal Analysis, "Fourier transform" include all signals that can be showed by adding variety of sinusoids together. Signal analysis requires enormous amount of calculation with complex numbers to find the frequency of the signal and to remove the annoying high frequency part. The formula to remove high frequency part, also called "filter", is showing$G(w) = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt$. In this equation, G(w) is a frequency spectrum of g(t). g(t) is a signal in the time-domain, and G(w) means varies in frequency, which is in the frequency-domain. In the field of oscillating springs, complex number is also an important factor. In fact, all the objects around us have oscillation that follow natural frequency. When an external object's frequency meets with the natural frequency of a whole system, energy can transport between, which may cause swaying of the amplitude of the system's oscillation. This phenomenon is called "resonance", and it is actually very dangerous. For example, when people walk across a bridge, the frequency of people walking may match to the natural frequency of that typical bridge. It will cause the whole bridge becomes unconsolidated and collapse finally. To prevent this from happen, dampeners need to be built to decrease the amplitude when energy is transferred. There is an equation that can express the decrease in amplitude in a mathematical and convenient way, $mx^2 + rx + k = 0$. In this equation, "m" represents mass in a system, "r" represents the resistive force, and "k" represents stiffness of the spring. To solve this equation, we can finally get solutions written in the form of "x=a+bi", which contain imaginary part. As a conclusion, complex numbers need to be used in multiple regions of science. It is also a new way for people to think about old or unsolved problems. Nowadays, there are more and more research about complex number to find more of its properties. It will become even more critical and useful in the future.

2. Historical development of complex variable

The development of complex number can be first chased back to 50 A.D. The Heron of Alexandria was wishing to calculate an impossible volume of a pyramid[1-3]. In his calculation, he derived $\sqrt{81-114}$ which cannot be shown as any real number solution. As a result, The Heron of Alexandria regard this as an impossible study and failed to solve the problem. Later on, the finding of solutions to cubic equations eventually lead mathematicians to the actual development of complex number.

People realise in the cubic equation, that there are two situations that either all three solutions are in real numbers, but also there is the existence of complex number solutions. Therefore, in order to find a general formula to solve the equation that Girolamo Cardano found a general method to solve complex number and published in his book called Ars Magma [4-6]. However, during the time, mathematicians were still very unfamiliar with the concept of complex number and disliked them that Cardano regarded complex number as "as subtle as it would be useless" and indicated working with those numbers were under "mental torture". In 1637, mathematician Rene Descartes invented the standard form of the complex number as a + bi and determines the part bi call as the term 'imaginary'. But other than his discoveries, Descartes still disliked the concept of complex number and referred to them as problems that were impossible to solve [1-10]. Along with Rene Descartes, Issac Newton and Albert Girad both indicated the impossible solution to complex number.

However, in 1572, Rafael Bombelli was the first to claim complex number and determine its usefulness. He published his book Algebra and included the basic arithmetic of complex number and the idea of complex conjugate ($a \pm bi$)) which his work is considered an important contribution in the development of complex number[1-10]. In 1685, John Wallis was the first man to come up with the idea of plotting complex number on the plane such that real number are plotted on the x-axis and imaginary numbers plotted on the y-axis.

Moving on, in 1777, Euler introduced the notation of $i = \sqrt{-1}$ and discovered the very famous Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$ which compressed five math constants and three operations in one equation [1-10]. In 1806, Jean Robert Argand elaborated on the thoughts from John Wallis. He demonstrated the geometric interpretation of complex number which is known as the Argand diagram named after his name. Finally, in 1831, Carl Friedrich Gauss provided the first clear understanding of complex number and the functions of it. He declared that there is at least one real or complex root for every polynomial with a real coefficient. Gauss put together the work made by past mathematicians and finally made the idea of complex number well-accepted. In 1826, Augustin Louis Cauchy came up with a formal definition of residue and he pioneered both real and complex study of analysis. Years later, In 1833, William Rowan Hamilton indicated complex number in pairs of real number ($3 + 4i$ as $(3,4)$) which made the idea of complex number easier to understand and easier to accept the concept.

In addition, many well-known mathematicians, such as Karl Weierstrass, Hermann Schwarz and Henri Poincare all studied in the general theory of complex number[1-10]. Furthermore, the sixteen years old Niels Abel discovered proof in extending Euler's binomial theorem to all numbers. The German mathematician August Ferdinand Mobius made further notes on the geometric interpretation of complex number and contribute his work in analytic geometry and topology[1-10]. By far to today, complex analysis had shown its significance and important use in both applied and pure mathematics. All of the mathematicians mentioned demonstrate the work been done in deriving the current knowledge of complex number and without them, we would not be able to apply that knowledge in real life.

3. Theorems of complex variable

3.1. General theorem

3.1.1. Cartesian and Polar representation of complex variable

In the first chapter we learn about complex numbers. We can show a complex number in a complex plane. In a complex plane, "x" axis shows the real part and "y" axis shows the imaginary part, so we can also show a complex number in a form of" z=(x,y)". We can find the sum of two complex numbers by using the formula "(x1,y1)+(x2,y2)=(x1+x2,y1+y2)" and find the product of two complex number by using the formula "(x1,y1)(x2,y2)=(x1x2-y1y2,y1x2+x1y2)". Also, every complex number have moduli, and we can find the moduli using the formula "$\sqrt{x^2 + y^2}$". Complex conjugate turns "x+iy" into "x-iy', which also means to let the complex number reflect over the real axis. And then, we learn about the polar form of a complex number "$z = r(\cos\theta + i\sin\theta)$" and the exponential form of a complex number "$z = re^{i\theta}$". After that, we start to learn about roots of complex number. In this chapter, we need to first imagine a complex number in a form of "$z = re^{i\theta}$" lies on a circle with radius "r". In this circle, as "θ" increase, the position "z" will move in counterclockwise direction. By understanding this, we can finally get a formula to find "n" roots of complex numbers, which is "$\sqrt[n]{r0}\exp\left[i\left(\frac{\theta 0}{n} + \frac{2k\pi}{n}\right)\right]$ (k=0, 1, 2, …, n-1)". The second topic we talk in the class is mainly about Cauchy-Riemann Equations, but we first learn about continuity, limit, derivative of a function. After we learn those chapters, we can first write an equation that contains complex number in a form of "$f(z) = u(x,y) + iv(x,y)$". By assuming that "$f'(z)$" exist for this equation, we can write that "$\Delta z = \Delta x = i\Delta y$". Finally, we can use this equation to prove that "$Ux(x0,y0) = Vy(x0,y0)$" and "$Uy(x0,y0) = -Vy(x0,y0)$" these two equations are necessarily true. It means that in this equation, "$Ux = Vy$", "$Uy = -Vx$", and "$f'(z0)$" can be written in a form of $f'(z0) = Ux + iVx$". In fact, the two equation that has been proved earlier are called Cauchy-Riemann Equations, found by French mathematician A.L.Cauchy.

3.1.2. Exponential function of complex number

The exponential function of complex number can be defined as:

$$e^z = e^{x+iy} \qquad (z = x + iy) \tag{1}$$

that according to Euler's formula

$$e^{it} = cos\boldsymbol{t} + isin\boldsymbol{t} \tag{2}$$

Where z is taken in value in radians. By applying the definition, when t = 0, which in calculus, e^z reduces to normal exponential function.

In addition, we can also apply definition (1) $e^z = re^{i\theta}$ $which$ $r = e^x$ and $\theta = y$, (3)

And this shows several related definition: $|e^z| = e^x \quad \arg(e^z) = y + 2n\pi \ (n \in Z)$ (4)

Moreover,

$$e^{z1}e^{z2} = e^{z1+z2} \tag{5}$$

$where\ (z1 = x1 + iy1 \text{ and } z2 = x2 + iy2) e^{z1}e^{z2} = (e^{x1}e^{iy1})(e^{x2}e^{iy2}) = e^{(x1+x2)}\ e^{i(y1+y2)}$. And by applying the same method, we can derive

$$\frac{e^{z1}}{e^{z2}} = e^{z1-z2} \tag{6}$$

Also, according to trigonometric period as $\sin(\theta + 2\pi) = \sin(\theta),\ \cos(\theta + 2\pi) = \cos(\theta)$. Therefore,

$$e^{z+2\pi i} = e^{z} \tag{7}$$

3.1.3. Definite integrals of function

We suppose $r(t)$ is a complex – valued function and t is a real value, which we can write as:

$$r(t) = u\,(t) + iv(t) \tag{8}$$

$$\int_a^b r(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt \tag{9}$$

And separate them as

$$Re\int_a^b r(t)dt = \int_a^b Re[u(t)]dt \quad and \quad Im\int_a^b v(t)dt \ = \int_a^b Im[r(t)]dt \tag{10}$$

3.1.4. CAUCHY – GOURSAT THEOREM

The Cauchy- Goursat theorem states the integral of the analytic function $f(z)$ around contour, both simple and closed, is zero.

Let C denotes contour z = z(t), which is a simple closed contour (a $\leq z \leq$ b) and we can write the integral of the analytic function f as $\int_C f(z) = \int_a^b f[z(t)]z'(t)\ dt$.

By letting:

$$f(z) = u(x, y) + iv(x, y) \text{ and } z(t) = x(t) + iy(t) \tag{11}$$

Then:

$$\int_C f(z)dz = \int_a^b (ux' - vy')dt + i\int_a^b (vx' + uy')dt \tag{12}$$

Therefore:

$$\int_C f(z)dz = \int_C u\,dx - vdy + i \int_C v\,dx + u\,dy \tag{13}$$

By applying the Green's theorem, we are able to express (13) in terms of double integrals whereas R is a closed region consisting all the point on the C and inside C. $\int_C f(z)dz = \iint_R (-v_x - u_y)\,dA + i\iint_R (u_x - v_y)\,dA$. Again, recalling the Cauchy-Riemann equations: $u_x = v_y$ and $u_y = -v_x$.

Hence, the integrals of the two double integrals are zero within R. Thus, while the function f is analytic and f ' is continuous in R $\int_C f(z)dz = 0$.

3.1.5. Cauchy Integral Formula

The formula states, if function f is analytic inside the simple closed contour C, and if z_0 is any point inside C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z - z_0} \tag{14}$$

Which indicates if f is analytic inside the simple closed contour C, then values of f inside to C are determined by the f on C.

This can be proved as letting C_p denote a circle, positively oriented, $|z - z_0| = \rho$ and C_p is inside C. Since $\frac{f(z)}{z - z_0}$ is analytic in and on the contour C and C_p, which according to the principle of deformation of paths, we can write

$$\int_C \frac{f(z)dz}{z - z_0} = \int_{C_p} \frac{f(z)dz}{z - z_0} \tag{15}$$

And continues writing as

$$\int_C \frac{f(z)dz}{z - z_0} - f(z_0)\int_{C_p} \frac{dz}{z - z_0} = \int_{C_p} \frac{f(z) - f(z_0)}{z - z_0} dz \tag{16}$$

Whereas we know $\int_{C_p} \frac{dz}{z - z_0} = 2\pi i$. So the equation can be rewrite as

$$\int_C \frac{f(z)dz}{z - z_0} - 2\pi i f(z_0) = \int_{C_p} \frac{f(z) - f(z_0)}{z - z_0} dz \tag{17}$$

We see that for f is analytic and continues. At z_0, which always corresponding to ϵ, a positive number such that

$$|f(z) - f(z_0)| < \epsilon \tag{18}$$

$$whenever\ |z - z_0| < \tau \tag{19}$$

By letting the radius ρ smaller than the number τ in (5) that it allows us to apply the theorem that C is a contour of length L and the function f(z) is continuous on C. M represents a non-negative constant which that

$$\left|\int_C f(z)dz\right| \leq ML \tag{20}$$

Now we can use (18) to derive $\left|\int_C \frac{f(z) - f(z_0)}{z - z_0} dz\right| < \frac{\epsilon}{\rho} 2\pi\rho = 2\pi\epsilon$ .By evaluating above, we derive $\left|\int_C \frac{f(z)dz}{z - z_0}\right| - 2\pi i f(z_0) = 0$. Hence $\left|\int_C \frac{f(z)dz}{z - z_0}\right| = 2\pi i f(z_0)$. Let Ω be an open set that contains a piece-wise smooth curve "r" with two end points "w_1" and "w_2". If "f" is continuous on Ω with primitive F, we can write that $\int_r f(z)\, dr = F(w_2) - F(w_1)$. A function f : $\Omega \rightarrow$ € is holomorphic at z_0 if and only if there exists a∈€ and r>0 so that far $|h|$< r: $f(z_0 + h) = f(z_0) + ah + h\ell(h)$, where ℓ: Dr $(0)|\{0\}$ → € satisfies ℓ(h)→0 as h→0. In this case, a=f'(z_0). If function f is holomorphic within a disc D, which f will have a primitive in D. We let D = $D_r(z0)$ and that z $\in D_r(z0)$. Let γ_z be a piece - wise smooth curve starting at $z0$ and end at z. Initially, γ_z travels in a horizontal direction from $z0$ to $Re(z) + Im(z0)$. Then, γ_z travels in a vertical direction from $Re(z) + Im(z0)$ to z. We then define F: $D_r(z0) \rightarrow \mathbb{C}$ as $F(z){:}\ \int_{\gamma_z} f(w)dw$

We state that F is a primitive of the function f in $D_r(z0)$. Now, let h to be small enough in which $z + h \in D_r(z0)$.To derive the difference between F(z+h) – F(z), we relate γ_{z+h} and γ_z.By observing the relations among curves, we see $F(z + h) - F(z) = \int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)\, dw = \int_p f(w)dw = \int_R f(w)dw + \int_T f(w)dw + \int_z f(w)dw$

In Gorsat's theorem, it states that $\int_R f(w)dw = \int_T f(w)dw$. Hence $F(z + h) - F(z) = \int_\xi f(w)dw$, where ξ is the line between z and z+h. And in such case, length(ξ) =h. $\int_\xi f(w)dw =$

$\int_{\xi} f(z)dw + \int_{\xi} f(w) - f(z)dw = hf(z) + \int_{\xi} f(w) - f(z)dw$. Therefore, $\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \left|\frac{1}{h}\int_{\xi} f(w) - f(z)dw\right| = \sup_{w\in\xi}|f(w) - f(z)|$. Since the function f us continues at z, we ensures the last expression will be small, by setting h to be small and therefore ξ is short. In conclusion, this shows F is holomorphic where F'(z) = f(z) and F is a primitive of f.

3.1.6. Cauchy theorem for a disc

The theorem states that if function *f* is holomorphic on a disc D, which we have the following on any closed and piece -wise smooth curve that $\gamma \in D$: $\int_{\gamma} f(z)\,dz = 0$.

3.2. Advanced theorem

3.2.1. Cauchy theorem in a disc

There are local primitives and Cauchy's theorem in a disc. The first theory is" in f is holomorphic on a disc d , then f has a primitive in d . If we want to proof this theory, we can first let $D = Dr(z0)$ and z belongs to Dr(z0). And then set a smooth curve r(z). r(z)will first travel horizontally, then travels vertically to point z. We can define it by the equation$F(z) := \int_{r(z)} f(w)dw$. In the equation, we let F be the primitive for f in Dr(z0). h should be small enough to make z+h belongs to Dr(z0). Also, by understand F(z+h)-F(z) and the relationship between r(z) and r(z+h), we can write the equation $F(z+h) - F(z) = \int_{r(z+h)} f(w)dw - \int_{r(z)} f(w)dw = \int_{p} f(w)dw = \int_{R} f(w)dw + \int_{T} f(w)dw + \int_{Z} f(w)dw$. By using the curve curves, Gorsat's theorem a tells that $\int_{R} f(w)dw = 0 = \int_{T} f(w)dw$.

So we can see that $F(z+h) - F(z) = \int_{z} f(w)dw$. Then we can use Theorem 3.1.9 to finally get $\left|\frac{F(z+h)-F(z)}{h} - f(z)\right| = \sup\text{ (w belong to z)}|f(w) - f(z)|$.

Since f is continuous at z, we can show that F is holomorphic at z with F'(z)=f(z), which means F is a primitive for f.

The Theorem 3.1.12 which is also the Cauchy theorem for a disc, is "if f is holomorphic in a disc D, then $\int_{r} f(z)dz = 0$.

Construct an open set Ω c € that containing circle C and interior disc D. We can write the equation $\int_{C} f(z)dz = 0$.

To prove it, we need first illustrate a disc satisfying C u D c D' c Ω, which is that same from the previous theorem with r=c. Then, let D=Dr(z0), if Dr(z0) is not contained in Ω, there exist a convergent subsequence (Zn(u))(u belong to w), say with limit z.

The inequality $|Wn(u) - z| \leq \frac{1}{n(w)} + |Zn(w) - Z|$ Shows that the subsequence (Zn(u))(u belong to w) converges to z. Because Dr(z0) and Ω are both closed, we can sure that z belongs to Dr(z0) ∩ Ω . So we choose D':=Dr+£(z0).

These hypotheses are necessary when we saw $\int_{Cr(0)} \frac{1}{z}dz = 2\pi i$

By using the methods above, we can perfectly prove these hypotheses.

For example, we can let r c € be a "keyhole" curve consisting of an exterior circle, an interior circle, and a narrow corridor joining them. The interior of r is the region r_{int} between the two circles but outside of the corridor. We also need to give r an orientation to ensure that r_{int} is on our left as we travel r.

Suppose f is holomorphic on an open set, we want to show $\int_{r} f(z)dz = 0$

And f should have a primitive on a neighborhood of r.

Since Ω is open, a larger keyhole region R containing r u r_{int} exist. Let p_z be a curve in R consist vertical and horizontal segment that start at z_0 and end at z, we can define it by $F(z) := \int_{p_z} f(w)dw$. And if we create another similar curve $\mathfrak{Z}_z$, then$\int_{p_z} f(w)dw - \int_{\mathfrak{Z}_z} f(w)dw$ can be computed in terms of integral over rectangular curves that contained R only with their interiors. By using Theorem 3.1.11, we can see that F is the desired primitive on R. The example give an exhaustive description of all such curves, but it can be transferred into a definition about "toy contour"

The definition of a toy contour is "a piece-wise smooth curve whose notion of interior is obvious and for which the construction of a primitive as in Theorem 3.1.11 is possible in a neighborhood of the curve and its interior. The positive orientation of such a curve is the one which keeps the interior on the left as one proceeds along the curve".

Construct a toy contour r with a interior r(int). If f is holomorphic on an open set Ω that contain r U r(int), we can sure that $\int_r f(z)dz = 0$

Here is a remark about Jordan curve theorem. The theory mainly says that if r c € is a simple, closed, smooth curve, then there exist two open set: one is bounded and the other one is unbound. In this theorem, the former is called the interior of r, and it is "simple connected". The latter is called the exterior of r.

For an example, we claim $\int_0^\infty \frac{1-\cos(x)}{x^2} dx = \frac{\pi}{2}$.

We can construct a function $f(z) := \frac{1-e^{iz}}{z^2}$.

that is holomorphic on an open set containing the indented semicircle $r_{£1R}$

In this situation, £ is bigger than 0 and smaller than R. Denote by $r_£$ and r_R the semicircle of radios £ and R, and with orientations compatible with $r_{£1R}$

Corollary 2.6 says that$0 = \int_{r_{£1R}} f(z)dz = \int_{-R}^{-£} f(x)dx + \int_{r_£} f(z)dz + \int_£^R f(x)dx + \int_{r_R} f(z)dz$.

It is an even function, so we can write$\mathrm{Re}[I1(£1R) + I3(£1R)] = 2\int_£^R \frac{1-\cos(x)}{x^2} dx$.

To prove the equation we claimed above, we need to show Re[I1(£1R)+I3(£1R)]→π when £→0 and R→∞.

If Im(z)≥0, we can write$|f(z)| \leq \frac{1+e^{\mathrm{Im}(z)}}{|z|^2} \leq \frac{2}{|z|^2}$. Furthermore, by using the power series expansion $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. We can write the function $f(z) =: \frac{-i}{z} + g(z)$.

In this function, $g(z) \to \frac{1}{2}$ as z→0. By using the parametrization $z(t) = -£e^{-it}$, $0 \leq t \leq \pi$, for $r_£$ we have$\lim_{£\to 0, R\to\infty} \mathrm{Re}[I1(£1R) + I3(£1R)] = \lim_{£\to 0, R\to\infty} \mathrm{Re}[-I2(£) - I4(R)] = \pi$

Which can finally prove the equation claimed.

3.2.2. Gorsat's theorem

3.2.2.1 Gorsat's theorem part 1

Let Ω belongs to complex number and it is an open set that contains a triangular curve T and T's interior. The theorem states: $\int_T f(z)dz = 0$.

When $f(z)$ is holomorphic in Ω.

Proof of the theorem:

We assume T is counterclockwise oriented as shown in diagram 1. We deduce that $T^{(0)} = T$.

Moreover, we denote $d^{(0)}$ and $p^{(0)}$ to the diameter and perimeter of $T^{(0)}$. By continually bisecting each side in T, we derive triangular curves $T_1^{(1)}, T_2^{(1)}, T_3^{(1)}$ and $T_y^{(1)}$.

He orientation on the interior edge cancels each other, which we now have:

$$\int_T f(z)dz + \int_{T_1^{(1)}} f(z)dz + \int_{T_2^{(1)}} f(z)dz + \int_{T_3^{(1)}} f(z)dz + \int_{T_y^{(1)}} f(z)dz \quad (21)$$

We now let $1 \leq j \leq 4$ to be the index that correspond to the largest of $\left|\int_{T_j^{(4)}} f(z)dz\right|$. we also denote that $T^{(1)} = T_j^{(4)}$. By applying the triangle inequality, we now have

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq 4\left|\int_{T^{(1)}} f(z)dz\right| \quad (22)$$

Now, we assign $d^{(1)}$ and $p^{(1)}$ to be the diameter and perimeter of $T^{(1)}$. Therefore, satisfying $d^{(1)} = \frac{1}{2}d^{(0)}$ and $p^{(1)} = \frac{1}{2}p^{(0)}$. And by repeating this step to $T^{(0)}, T^{(1)}, \dots, T^{(n)}$ that satisfies the condition in (3).

Thus, $T^{(n)'}$s diameter and perimeter is shown by $d^{(n)}$ and $p^{(n)}$. Also, we let $T^{(n)}$ to be the union that includes both $T^{(n)}$ and its enclosed area. Which, $T^{(n)}$ is now compact with diameter of ($T^{(0)} = \frac{1}{2^n}d^{(0)}$) and

$$T^{(0)} \supset T^{(1)} \supset \cdots \supset T^{(n)} \quad (23)$$

Hence, since there is has a unique point z_0 where it belongs to complex number. With z_0 also belongs to $T^{(n)}$ for all n belongs to N. We know that f is holomorphic at z_0, thus, by applying Lemma 3.1.10, we can find r > 0 which for Z belongs to $Dr(z_0)$, that we now have

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\gamma(z) \quad (24)$$

Where $\gamma: Dr(z_0) \to \mathbb{C}$ and also satisfies $\gamma(z) \to 0$ as z_0 approaches z. by Corollary 1.15, now have

$$\int_{T^{(4)}} f(z)dz + \int_{T^{(1)}} (z - z_0)\gamma(z)\, dz \quad (25)$$

where the constant function of $f(z_0)$ and the linear function of $f'(z_0)(z - z_0)$ both have primitives and $T^{(n)} \subset Dr(z_0)$. By letting $n \in N$ be large enough, which we can guarantee that $d^{(n)} \leq r$. under such scanerio, we can estimate this integral using proposition 1.13.3 :

$$\left|\int_{T^{(n)}} f(z)dz\right| \leq d^{(n)}p^{(n)} \frac{\sup}{z \leftarrow T^{(n)}} |\gamma(z)| = \frac{1}{\gamma^n} d^{(0)}p^{(0)} \frac{\sup}{z \leftarrow T^{(n)}} |\gamma(z)| \quad (26)$$

Now, we can use (3) to obtain

$$\left|\int_{T^{(0)}} f(z)dz\right| \leq d^{(0)}p^{(0)} \frac{\sup}{z \leftarrow T^{(n)}} |\gamma(z)| \quad (27)$$

And as $z \to z_0$, we have $\gamma(z) \to 0$. So the sequence on the right hand side will tends to zero as n approaches infinity. Hence the whole integral must tends to 0 as stated in (27).

3.2.2.2 Gorsat's theorem part 2

We now let $\Omega \in \mathbb{C}$ and it is an open set that contains a rectangular curve R and R's interior. The theorem states:

$$\int_T f(z)dz = 0 \quad (28)$$

With f is holomorphic anywhere in and on Ω.

Proof:

We let R to be oriented in a counterclockwise direction and divide R into two rectangles

Then, we note bisecting edge cancels each other and by integrate along it, we derive:

$$\int_R f(z)dz = \int_{T_1} f(z)dz + \int_{T_2} f(z)dz = 0 \tag{29}$$

Moreover, this arguments may adapt to any polygon since we can divide it into triangles and then apply the main theorem 1. Hence, these result will be the special cases of Cauchy's theorem that states $\int_\delta f(z)dz = 0$.

That function f is holomorphic anywhere on the open set δ and its enclosed area.

3.2.2.3 Using complex-valued knowledge to solve a real value problem

Theorem:

$$\forall \boldsymbol{n} \in \boldsymbol{N}^*, \mathbf{sin\,1} + \mathbf{sin\,2} + \boldsymbol{sin3} + \cdots + \mathbf{sin}\,\boldsymbol{n} \le \frac{\mathbf{1}}{\boldsymbol{sin}\frac{\mathbf{1}}{\mathbf{2}}} \tag{30}$$

Proof

By Euler's formula, we know

$$\cos\theta + i\sin\theta = e^{i\theta} \tag{31}$$

So, matching the component, we see

$$\sin\theta = \mathrm{Im}\left(e^{i\theta}\right) \tag{32}$$

Thus, we can write the RHS of (1) as

$$\sin 1 + \sin 2 + \sin 3 + \cdots + \sin n = \mathrm{Im}\left(e^{i} + e^{2i} + e^{3i} + \cdots + e^{ni}\right) \tag{33}$$

And the LHS of (4) can be simplified as

$$= \mathrm{Im}\left(\frac{e^{i}(1-e^{in})}{1-e^{i}}\right) \tag{34}$$

further expanding the bracket of (35) we now have

$$= \mathrm{Im}\left(\frac{e^{i}-e^{i(n+1)}}{1-e^{i}}\right) \tag{35}$$

Now, substitute the values from (2) we have

$$= \frac{\cos 1 + i\sin 1 - \cos(n+1) - i\sin(n+1)}{1-(\cos 1 + i\sin 1)} \tag{36}$$

Which we can simplify this by arranging similar terms together and derive:

$$= \frac{[(\cos 1 - \cos(n+1)) + i(\sin 1 + \sin(n+1))](1-\cos 1 + i\sin 1)}{(1-\cos 1 - i\sin 1)(1-\cos 1 + i\sin 1)} \tag{37}$$

By only considering the imaginative part, we may ignore the real parts of (8) and left with

$$\mathrm{Im}\left(\frac{e^{i}(1-e^{in})}{1-e^{i}}\right) = \frac{\sin 1(\cos 1 - \cos(n+1)) + (1-\cos 1)(\sin 1 - \sin(n+1))}{2-2\cos 1} \tag{38}$$

Using the compound angle formula, we see

$$\sin 1 = 2\sin\frac{1}{2}\cos\frac{1}{2} \quad \& \quad 1-\cos 1 = 2\sin^2\frac{1}{2} \tag{39}$$

Hence, substitute values in (10) back (9) we now have

$$\frac{\cos\frac{1}{2}(\cos 1 - \cos(n+1)) + \sin\frac{1}{2}(\sin 1 - \sin(n+1))}{2\sin\frac{1}{2}} \tag{40}$$

By using the compound angle formula again, we derive the value

$$\cos\frac{1}{2}\cos 1 + \sin\frac{1}{2}\sin 1 = \cos\frac{1}{2} \tag{41}$$

&

$$\cos\frac{1}{2}\cos(\mathrm{n}+1)+\sin\frac{1}{2}\sin(\mathrm{n}+1)=\cos\left(\mathrm{n}+\frac{1}{2}\right) \quad (42)$$

And further simplify this expression by using the values in (12), we derive

$$\frac{\cos\frac{1}{2}-\cos\left(\mathrm{n}+\frac{1}{2}\right)}{2\sin\frac{1}{2}} \quad (43)$$

Since we know that $-1 \leq -\cos\left(\mathrm{n}+\frac{1}{2}\right) \leq 1$,

Thus

$$\cos\frac{1}{2}-\cos\left(\mathrm{n}+\frac{1}{2}\right) \leq 2 \quad (44)$$

Therefore, we can have that

$$\sin 1+\sin 2+\sin 3+\cdots+\sin \mathrm{n} \leq \frac{2}{2\sin\frac{1}{2}} \quad (45)$$

Finally, we proved

$$\sin 1+\sin 2+\sin 3+\cdots+\sin \mathrm{n} \leq \frac{1}{\sin\frac{1}{2}} \quad (46)$$

4. Conclusion

The Cauchy theorem, a fundamental theorem in Complex analysis, can compute some real integrals that the fundamental theorem of Calculus cannot. We shall discuss how to compute real integrals using the Residue theorem in the future. Also covered will be the verification of Jensen's formula, growth order, and infinite products. It's crucial to understand how these theorems and concepts are connected to reveal some intriguing aspects of the complete function. The Hadamard factorisation theorem and Weierstrass infinite products go deeper into function theory, where many intriguing structures can be found.

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