Complex Analysis and its Several Applications

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Abstract. Complex analysis is important for science because it extends analytical methods from real variables to complex variables and complex numbers. Also, complex number has two independent components, one variable will not change when the other is changing, that are particularly useful when two variables must be dealt with simultaneously. In this essay, we are going to talk about the history chronologically such as who first introduced the idea of complex number, who first discovered the rule of complex number, and why complex analysis is important. Also, the essay includes some basics about the complex variable and complex analysis. For example, the definition of complex number, Cauchy-Riemann Equations, and Cauchy Goursat theorem can help us to get further known of the complex analysis and solve some basic analytic problems.

Keywords: Complex analysis, Complex number

1. Introduction

Mathematics is a very abstract subject, especially the function of complex variable. It is difficult imagine that how function of complex variable connects with practice, since it is very sophisticated, uses some intricate theorem, and includes imaginary number that hardly appear in real life. However, it is actually everywhere. For example, physics is one of the closest subjects about the complex variable function. The current in our daily life is three-phase. We can use imaginary numbers to deal with the phase angle relationship through the RCL circuit. It reveals some physical characteristics of imaginary numbers in a certain sense. In addition, a well-known hypothesis, Riemann hypothesis, also contain function of complex variable. So, back to the beginning. What is complex numbers and complex analysis? A complex number is a number of the form $a + bi$, where a and b are real numbers, and i is an indeterminate satisfying $i^2 = -1$. For example, $4 - 7i$ is complex number. Moreover, the study of functions of a complex variable is known as complex analysis.

Although historically we have named it “imaginary,” complex numbers are considered as “real” as real numbers in mathematical science, and are very fundamental to many scientific interpretations of the real world, from analyzing the solution of polynomial equations, algebraic characterization and circle to solving engineering problems, explaining quantum mechanics, to analyzing periodic signals. Complex number and complex analysis can help us to solve problems that seem no conclusion and complicated. Also take current as an example. If the phase of the current is calculated by trigonometric function, it is more complex and abstract, and many engineering problems, such as impedance, alternating electricity, and oscillating mechanical system, cannot be solved. Instead, if we combine real number and imaginary number together to figure out the question, we can derive the answer successfully and skilfully. In conclusion, using complex number can change an intricate puzzle to into simple one. That is the reason we use complex number to solve problems and it is important for our modern science.

Complex numbers have applications in many scientific fields, including signal processing, electromagnetism, fluid dynamics, and quantum mechanics. I am going to list three of the most important application of complex number and complex analysis.

The first is system analysis. Systems are often transformed from the time domain to the frequency domain by the Laplace transform. So, the poles and zeros of the system can be analyzed on the complex plane. The root locus method, Nyquist plot and Nichols plot are used to analyze the stability of the system on the complex plane. Whether the poles and zeros of the system are in the left half plane or the right half plane, the root locus method is very important. If the pole of the system is in the right half plane, the causal system is unstable. If both are located in the left half plane, the causal system is stable. On the virtual axis, the system is critically stable. If all the zeros and poles of the system are in the left half plane, it is a minimum phase system. A system is all-pass if its poles and zeros are symmetric about the imaginary axis [1].

The second is Quantum Mechanics. Complex numbers are important in quantum mechanics because their theory is based on infinite dimensional Hilbert Spaces over complex fields. Some space-time Metric equations in special and general relativity can be simplified by treating the time variable as an imaginary number. In the practical application of applied mathematics, to solve the system of a given difference equation model, we usually first find out all the complex characteristic roots r of the characteristic equations corresponding to the linear difference equation, and then express the system as a linear combination of basic functions of form F (t) = E [1].

The third is signal analysis. Signal analysis and other fields use complex numbers to easily represent periodic signals. Die | z | value signal amplitude, argument arg (z) of a given frequency sine wave phase. The real signal can be expressed as the sum of a series of periodic functions by Fourier transform. These periodic functions are usually expressed as the real parts of complex functions of the form: where ω corresponds to angular frequency, and the complex number Z contains information about amplitude and phase [1].

2. History of complex number

2.1. History development

Solution of the cubic $x^3 = px + q$ was developed in the Renaissance by Italian mathematicians. Three mathematicians Scipione del Ferro and Niccolò Tartaglia, followed by Girolamo Cardano, had proved that $x^3 = px + q$ has a root as $x = \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$-$\sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{p^3}{27}}}$. First introduced by G. Cardano in 1545[1-4] He was trying to use complex number as a tool for finding real roots of a cubic equation ($x^3 + ax + b = 0$). However, he was highly confused about the expressions (e.g., 5+$\sqrt{-15}$ and 5-$\sqrt{-15}$). Since adding them we can obtain 10, while multiplying them we will get 40. What the two expressions stands for that we can divide 10 in two parts, the product of which is 40. And Cardano considered thinking about them as "mental torture"

In 1572, R. Bombelli introduced the symbol and the calculating rules for complex analysis [4-7]. To be more specific of what he had done, he considered the equation $x^3 = 15x + 4$ as an example. It can be clearly observed that the equation contains a root of 4. And Bombelli applied Cardan Formula to get the result x=$\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$. Although the two function seems to be utterly different from each other, Bombelli try to prove it is right by setting $\sqrt[3]{2+\sqrt{-121}}$ = a+bi. Thus, he was able to deduce that$\sqrt[3]{2-\sqrt{-121}}$=a-bi. After a series of algebraic manipulations, he got a=2 and b=1, which gives x the same result as 4. This was an astonishing result as it proved the contribution of complex number in solving cubic equation.

In 1730, Abraham de Moivre introduce de Moivre's theorem that $(\cos(\theta)+i\sin(\theta)^n=\cos(n\theta)+i\sin(n\theta)$ where n is an integer. And this appear in Cardan formulas also, when there is irreducible case to be solved [7-9].

Leonard Euler have introduced a number of notations and formula that are used frequently[1-10]. First, he establishes the notation: i=$\sqrt{-1}$ and mentioned that it is impossible to compare the square root of a negative number with a possible number (real numbers). Thus, from the idea of numbers (all numbers are possible to conceive are either greater or less than 0, or are 0 itself), the nature of complex numbers is impossible. So, Leonard Euler considered them as imaginary quantities, since they exist only in the imagination. He states the major usage of imaginary numbers are in calculation. Leonard Euler was also the founder of formulas: x=iy = r（$\cos(\theta)+i\sin(\theta)$） and $e^{i\theta}$= cos(θ)+ isin(θ). He also gives a visualized model (a regular polygon with n side) for $z^n=1$ by applying x=iy = r（$\cos(\theta)+i\sin(\theta)$).

Carl Friedrich Gauss established a+bi as "complex number" and he managed to give four proofs of the fundamental theorem of algebra, including the one that any nth-degree polynomial has n roots, some or all of which may be imaginary. In 1833, William Rowan Hamilton introduced the first rigorous definition of complex number that a complex number a+ib can be seen in the form (a, b). Following defined the addition and the multiplication of the complex couples as (a, b)+(c, d) = (a + c, b + d), since a+ib + c+id= a+c+i(b+d), And (a, b)(c, d) = (ac-bd, bc+ad), since (a+ib)(c+id)= ac+iad+ibc-bd = ac-bd+i(ad+bc) [6].

Jean-Robert Argand establish the argand diagram in 1806, which used to provide a imagine representation for complex numbers. Turning to complex analysis, the first one start to develop the theory of the complex integral calculus was Augustin-Louis Cauchy. He employed imaginary numbers to evaluate "real integrals" like $\int_0^\infty \frac{\sin x}{x}$ dx and $\int_0^\pi \log\sin x\ dx$ which was impossible to be evaluated previously and obtain astonishing results $\int_0^\infty \frac{\sin x}{x}$ dx = $\frac{\pi}{2}$ and $\int_0^\pi \log\sin x\ dx=-\pi\log 2$ [1-10].

2.2. Idea of complex analysis

We were first introduced to the foundation idea of complex analysis：there are three forms of complex number: Cartesian form,

$z=x+iy$, Polar form, $z=\rho e^{i\theta}$, and Euler's formula $e^{ix}=\cos(x)+i\sin(x)$. We also learnt about the fundamental complex multiplication and addition that similar to the normal addition and multiplication, along with the introduction of complex conjugation during the first lecture. If $z=x+iy$, the complex conjugate of z is $\bar{z}=x-iy$. Within the calculation of complex number with its complex conjugate, we sometimes use formula for the difference of square to help us solve the equations. However, there is some problems when we use polar form to represent complex number. There are infinite number of values of θ, but we cannot use all of them to represent a complex number. So, we decide to use θ between 0 and 2π.

The idea of complex plane and complex function was also taught, which enable the further learning of how to judge whether a function is differentiable by using Cauchy Riemann condition. We also learnt how to integrate a function along a path, and unlike the integration of real number, complex integration doesn't contain any geometrical significance.

Following, several theorems: Cauchy Goursat theorem, Cauchy's integral formula, Taylor's formula and Laurent expansion and Cauchy's residue theorem which are extremely important for complex integration were taught.

Cauchy Goursat theorem state that if f(z) is holomorphic or differentiable at every point in U, ($z_0 \in$ U) $\int_\gamma f(z)dz = 0$. It is hard to believe because even in the case when γ in the boundary of a rectangle contained in U, it is true. We can use this theorem to estimate the integral value of a closed differentiable function.

Cauchy' integral formula is $f(z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0}dz = 0$, if f(z) is holomorphic in an open region that include the closed disc. Also, we can complete $f(z_0)$ using only value of f(z) on the circle.

Cauchy residual theorem is that f(z) is differentiable at every point in U except at points $z_1, z_2, z_3 \dots \dots z_p$. For example, if a function is met the condition of the Cauchy residual theorem, we first rewrite its denominator and split it into two separate part and identity whether these points are in the circle or not. And finally adding the residul of the point inside the circle.[2]

For the final project, we were assigned with a complex function with denominator Z^4-1 and numerator $10Z^3$+(-2+2i)Z^2-2Z-(2+2i). In order to integrate this function using Cauchy's residue theorem, we have to change the denominator into Z- C form (C is a constant). So we factorize Z^4-1 into (Z-1)(Z+1)(Z-i)(Z+i) and use algebra to figure out the numerator for each denominator after splitting the origin function into four. Then we can extract the result for different loci by drawing the picture and identifying which value of Z (1, -1, i, -i) was included. Finally, according to Cauchy's residue theorem, we can get the result.

2.2.1. Complex number and complex plane

Definition of complex number is the complex numbers are the set $\{x + iy: x, y \in R\}$, with $i = \sqrt{-1}$. Also, we define real part of complex number as $Re(z) := x$, and imaginary part as $Im(z) := y$. Another important terminology is called absolute value, or modulus, which write as $|z|$ and defines as $\sqrt{x^2 + y^2}$. Just as same as the real number, complex number can also do some calculation, such as addition and multiplication. First, the addition of complex number is very simple. Expanding the equation and the sum of real number is the new real part and the sum of the imaginary number is the new imaginary part. The general equation is $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$. The multiplication of complex numbers is similar to multiplication of binary equations. With the rule that $i = \sqrt{-1}$, multiplying each part and adding together. The general equation is $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$. Moreover, there is a unique operation of complex number, conjugation. We express it by adding a bar over the complex number, like $\overline{x + iy}$. This is equals to $x -$ iy. For another, we can use these basic rules and operations to conduct some other common equations, such as $\overline{z + w} = \bar{z} + \bar{w}$, $\overline{zw} = \bar{z} * \bar{w}$, $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$, and so on [1].

However, complex number is still intangible for us to understand. We are able to use complex plane to visualize complex numbers. Complex plane is akin to cartesian coordinate system but its x-axis is real part and y-axis is imaginary part. In this case, the x-axis is called the real axis, the y-axis is called imaginary axis, and the coordinate plane is called the complex plane. By recalling the distance formula for real number, we see that modulus of z is the distance from z to origin in the complex plane. More generally, the modulus of z-w, or $|z - w|$, is the distance from z to w. Given that coordinates can be rewritten into polar forms, complex number can also be written as polar form and translate this for complex number: for any z, $\left|\frac{z}{|z|}\right| = |z| * \frac{1}{|z|} = 1$. Thus, there exist $\theta \in R$, so that $\frac{z}{|z|} = \cos\theta + i\sin\theta$. If we write $e^{i\theta} = \cos\theta + i\sin\theta$, then $z = |z|e^{i\theta}$. We call this polar form of z. The angle θ is called argument of z and is denoted arg z.

2.2.2. Functions on the complex plane

We usually consider functions' continuity, and differentiability. First, to define continuity, let $\Omega \in C$, we say a function $f: \Omega \rightarrow C$ is continuous at $z_0 \in \Omega$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that whatever $z \in \Omega$ satisfies $|z - z_0| < \delta$. Then one has $|f(z) - f(z_0)| < \varepsilon$. We say f is continuous on Ω if it is continuous at every $z_0 \in \Omega$ [2].

Second, to define differentiability, we analogize this differentiability of real function. the definition of a function that is differentiable at $x = a$ if $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists. Thus, the definition of a complex function that is differentiable at $z_0 \in \Omega$ if $\lim_{C \to h \to 0} \frac{f(z_0+h) - f(z_0)}{h}$. Furthermore, if there exists $\omega \in C$, so that all $\varepsilon > 0$ there exists $\delta > 0$ so that whatever $h \in C \backslash \{0\}$ satisfies $|h| < \delta$. Then it has $z_0 + h \in \Omega$ and $\left|\frac{f(a+h) - f(a)}{h} - \omega\right| < \varepsilon$. In this case, we call ω the derivative of f at z_0 and write $f(z_0) := \omega$. We say f is holomorphic on Ω if it is holomorphic at every $z_0 \in \Omega$. We say f is entire if it is holomorphic on C.

Then, we can prove Cauchy-Riemann Equations. Given $f: \Omega \to C$, one can always define real-value functions. $u(x, y) := Re(f(x + iy))$ and $v(x, y) := Im(f(x + iy))$. So that $f(x + iy) = u(x, y) + iv(x, y)$. Suppose f is holomorphic at $z_0 = x_0 + iy_0$. For the moment, we will view $f(x, y) := f(x + iy)$ as a function on real number so that it makes sense to consider its partial derivatives. We can relate that to $f'(z_0)$ by considering different paths $C \to h \to 0$. First consider $R \to h \to 0$: $f'(z_0) = \frac{\partial f}{\partial x}(x_0, y_0)$. Next by considering $iR \to ih \to 0$: $f'(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(x_0, y_0)$. Therefore, we must have $\frac{\partial f}{\partial x}(x_0, y_0) = \frac{1}{i}\frac{\partial f}{\partial y}(x_0, y_0)$. Finally, by substituting $f = u + iv$ and taking real and imaginary parts, we obtain the Cauchy-Riemann equations: $\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$ and $\frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0)$. Thus, if f is holomorphic on an open set $\Omega \in C$ we have shown that its real and imaginary parts have partial derivatives satisfying the Cauchy-Riemann equations on all Ω. Furthermore, since holomorphic functions are infinitely differentiable, these partial derivatives will be continuous [3].

Third, to define analytic. For $f: \Omega \to C$, we way f is analytic on set Ω if there exists a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ with positive radius of convergence R>0 so that $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ for all z in a neighborhood of z_0. We say f is analytic on Ω if it is analytic at all $z_0 \in \Omega$.[4]

First, we define what is a smooth curve and what is a piece-wise smooth curve. We say $r < C$ is a smooth curve if there exists a function $z: [z, b] \to C$ such that $z([z, b]) = r$ and $z'([z, b]) \to C$ exists and is continuous here $[z, b] < R$ and $z'(t) = \lim_{h \to 0} \frac{z(t+h) - z(t)}{h}$ for $a < t < b$. We say r is a piece-wise smooth curve if there exists a continuous function $z: [z, b] \to C$ and points $a = a_0 < a_1 < \cdots < a_n := b$. The function z is called a smooth curve parametrization of r. We say r starts at z(z) and ends at z(b). We say r is closed if $z(a) = z(b)$, and say r is simple if $z|_{[a,b)}$ and $z|_{(a,b]}$ are injective.

2.2.3. Example

For $w \in \mathbb{C}$ and $n \in N$, the nth roots of w are the solutions z of the equation $z^n = w$

Show that $w \neq 0$ has exactly nth roots and find a formula for them.

Compute the square and cube roots of i

Suppose z is an nth root of 1 (called an nth root of unity)

Show that $1+z+\ldots+z^{n-1} = \begin{cases} n & \text{if } z = 1 \\ 0 & \text{otherwise} \end{cases}$

A, let w be $r(\cos(n\theta + 2k\pi) + i\sin(n\theta + 2k\pi))$, which k is a non-negative integer. According to the formula found by Abraham de Moivre that $(r^{\frac{1}{n}}(\cos(\theta) + i\sin(\theta)))^n = r(\cos(n\theta) + i\sin(n\theta))$, we can deduce that $r(\cos(n\theta + 2k\pi) + i\sin(n\theta + 2k\pi))= (r^{\frac{1}{n}}(\cos\left(\theta + \frac{2k\pi}{n}\right) + i\sin\left(\theta + \frac{2k\pi}{n}\right)))^n$. Since $\theta = \theta + 2\pi$, $\cos(\theta) = \cos\left(\theta + \frac{2n\pi}{n}\right)$, so only when $n > k \geq 0$, there exist a distinct root. Thus, the number of roots would be $n - 1 - 0 + 1 = n$,which means there are nth distinct roots exist. And the general solution formula for $z^n = w$ is $r^{\frac{1}{n}}(\cos\left(\theta + \frac{2k\pi}{n}\right) + i\sin\left(\theta + \frac{2k\pi}{n}\right))$ for w= $r(\cos(n\theta + 2k\pi) + i\sin(n\theta + 2k\pi)$, when k is in range $n > k \geq 0$.

B, first, according to the formula found by Euler that x+iy = r（$\cos(\theta) + i\sin(\theta)$, we can rewrite i into form $\cos\left(\frac{1}{2}\pi\right) + i\sin\left(\frac{1}{2}\pi\right)$. Following, according to the general formula obtained in a part. For square roots, which n=2, k can only be 0 and 1. Thus the square roots for i are $\cos\left(\frac{1}{4}\pi\right) + i\sin\left(\frac{1}{4}\pi\right)$ which can be simplified to $\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, and $\cos\left(\frac{5}{4}\pi\right) + i\sin\left(\frac{5}{4}\pi\right)$ which can be simplified to $(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2})$. And the cube roots of i, roots $\cos\left(\frac{1}{6}\pi\right) + i\sin\left(\frac{1}{6}\pi\right)$, $\cos\left(\frac{5}{6}\pi\right) + i\sin\left(\frac{5}{6}\pi\right)$ and $\cos\left(\frac{9}{6}\pi\right) + i\sin\left(\frac{9}{6}\pi\right)$ when applying k=0, k=1 and k=2 respectively to the formula.

C, If Z =1, then all power of Z equals to 1, so add n number of 1 together we will get n. If n is not 1, then we can express z as $\cos\left(\frac{2k}{n}\pi\right) + i\sin\left(\frac{2k}{n}\pi\right)$ as 1 can be written as $\cos(2k\pi) + i\sin(2k\pi)$. So Z^2, $Z^3 \ldots Z^{n-1}$, can be written as $\cos\left(\frac{4k}{n}\pi\right) + i\sin\left(\frac{4k}{n}\pi\right)$, $\cos\left(\frac{6k}{n}\pi\right) + i\sin\left(\frac{6k}{n}\pi\right) \ldots \cos\left(\frac{(2n-2)k}{n}\pi\right) + i\sin\left(\frac{(2n-2)k}{n}\pi\right)$ respectively. Thus, we can rewrite the function and since $\cos(x) + \cos(\pi - x)=0$ and $\sin(x)+\sin(2\pi - x)=0$. So $\mathrm{Re}(1 + z + z^2 + \cdots + z^{n-1})=1+ \cos\left(\frac{2k}{n}\pi\right)+\cos\left(\frac{(n-2)k}{n}\pi\right)+\cos\left(\frac{4k}{n}\pi\right) + \cos\left(\frac{(n-4)k}{n}\pi\right)+\ldots + \cos(\pi)+ \cos(\frac{(n+2)k}{n}\pi)+ \cos(\frac{(2n-2)k}{n}\pi)+ \cos(\frac{(n+4)k}{n}\pi)+ \cos(\frac{(2n-4)k}{n}\pi)+\ldots = 1+ \cos(\pi)=0$

And $\mathrm{Im}(1 + z + z^2 + \cdots + z^{n-1})=\sin(\frac{2k}{n}\pi)+ \sin(\frac{(2n-2)k}{n}\pi)+ \sin(\frac{4k}{n}\pi)+ \sin(\frac{(2n-4)k}{n}\pi)+\ldots=0$

So, if $z\neq 0$,so only possible sum is 0.

3. Conclusion

In conclusion, in this essay, we investigate the complex analysis from its application and its history, together we state some basic theorems of complex analysis and their founders. Additionally, we introduce of complex plane and its function. After the information section, we solve a question by applying the theorems previously introduced.

Following we are going to research more about the integration of section within the complex plane and the proof and application of practical theorems like Cauchy Goursat theorem, Cauchy's integral formula, Taylor's formula and Laurent expansion and Cauchy's residue theorem.

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