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A Fascinating Expedition of Complex Analysis

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Abstract. The invention of complex numbers was essential in the rapid development of mathematics. Although most people consider complex numbers to be nonsensical notations, they are critical in technical disciplines. Human beings entered a wonderful realm when mathematicians began to investigate complex numbers. Our universe is amazing, full of beautiful discoveries, and even miraculous, according to complex analysis. We begin by giving a brief review of complex analysis' history, including when and how it originated. The root of a quadratic equation was used to find complex numbers. We look at some of the ways complex numbers are used in engineering domains like acoustic wave propagation. A thorough examination of certain key theorems of complex analysis is also included.

Keywords: Complex analysis, Complex numbers

1. Introduction

Complex analysis uses varies techniques that are applicable in numerous fields. It lies on the intersection of several areas of mathematics, both pure and applied, and has important connections to asymptotic, harmonic and numerical analysis. When solving problems related to physics in real lives, complex analysis can always simplify complicated situations, building a model and analyzing with related theorems.

Propagation of acoustic waves relevant for the design of jet engines is strongly related to the boundary-integral techniques of complex analysis. Solid and fluid mechanics as well as conformal geometry in imaging, shape analysis and computer vision also include boundary-integral techniques into their syllabuses [1-10].

"Unified transform method of Fokas with its relation to the spectral theory and integrable systems" has its relation with Riemann-Hilbert problems with orthogonal polynomials and random matrices. Fre-bounded problems are widely studied in recent years driven by the spplication in mathematics, statistics and physics [1-10]. In finite-time singularity formation in various PDE problems, complex-domain plays important parts, overlaping fields of computing and PDE [1-10].

In recent times, there are also some function theory research work is no longer considers individual functions, but a family of functions with certain properties together to study. In fact, P. Montel's regular family of analytic functions should belong to this type of research and has shown its power. Studies from this point of view have developed considerably [10-11]. For example, Hp spaces, which have developed closer ties with other branches of mathematics. The extension of the theory of functions of complex variables from one variable to several variables is a very natural idea, always called complex analysis. However, the complexity of the domain of definition is greatly increased in the case of multiple variables, and the properties of the function are significantly different than in the case of single variables, and its study requires more recent mathematical tools.

The theory of functions of complex variables has a history of 150 years from the time of Cauchy. It has become an important part of mathematics with its perfect theory and excellent skills. It has advanced a number of disciplines and has often been used as a powerful tool in practical problems. Its basic content has become a required course for many majors in science and technology. The theory of functions of complex variables still has a number of topics that have yet to be studied, so it will continue to evolve and will achieve more applications.

The theory has been applied to many branches of mathematics, including fluid mechanics, stable plane length, aerodynamics, and other disciplines in physics. The theory of functions of complex variables has penetrated into calculus equations, number theory and other disciplines and has influenced their development. Nowadays, for example, Russian scientist used the theory of complex functions to solve the structural problems of airplane wings when designing airplanes, and he also made contributions to the use of the theory of complex functions to solve problems in fluid mechanics and aeromechanics.

Complex variables is a very important part of mathematics with its perfect theory and excellent skills. It has promoted the development of many disciplines, and is also a powerful tool in solving some practical problems.

The concept of complex numbers originates from solving the roots of a system of equations. As early as the middle of the 16th century, an Italian scientist in 1545 when solving three equations, firstly emerged with the idea of taking the square root of complex numbers. 17th century to the 18th century, the complex number began to have a geometric interpretation, it corresponds to the plane vector to solve practical problems. It arose in the 18th century, and was made by Euler[1]. Then it was fully developed in the 19th century. In the 20th century, complex functions were widely used in theoretical physics, elastic physics, and celestial mechanics. At the same time, complex functions are one of the earliest and most fruitful branches of mathematics in China.

The study of multivariable function theory, as early as in the single complex function theory of (G. F.) B. Riemann[2] and K. (T. W.) outside the era of Struthers has begun piecemeal. But it was the work of (J.) H. Poincaré[3], F. M. Hartogs[4] and others in the late 19th and early 20th centuries that really marked the creation of the discipline of the theory of functions of many complex variables. Their work revealed the intrinsic uniqueness of multivariate all-pure functions. Among them, two questions named on the overall properties of all-pure functions and the question of whether the proposed convex domain and the all-pure domain are equivalent, raised by E.E. Levy, had a profound influence and became a driving factor in the development of the theory of functions of many complex variables for a long time. In the 1930s, although there were important works such as K. Reinhardt's[5] work on analytic self-homogeneous groups. From the 1930s onward, the study of multiple complex variables saw an initial boom. In this period, there were outstanding results uniqueness theorem on totally pure self-homology, the properties of Lie groups of totally pure self-homology groups with bounded domains, and the Kadam-Sullen theorem on the equivalence of totally pure domains and totally pure convexity. In particular, starting from 1936, the Japanese mathematician conducted a long, systematic and fruitful study of the central problems of multiple complex variables, such as the Cushing problem, the Levy problem and the approximation problem, and finally gave answers to these problems in the 1950s. His series of works had a great influence on the development of multivariable in the later years.

After the 1950s, in line with the general trend of synthesis and abstraction in modern mathematics, the tendency to use topological and geometric methods to study the overall properties of all-pure

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functions in the theory of functions of multiple complex variables became more and more obvious. The concept of layers and their upper cohomology, introduced by J. Leray[6] into topology, was rapidly and successfully used in multiplicative variables. H. Glauert[7] solved the Levi problem for complex manifolds, and he, and others also developed the theory of complex spaces considerably. The whole 1950s was undoubtedly a golden age for the development of multiple complex variables.

Cauchy, Riemann, Weierstrass and others did a lot of groundwork for the development of this discipline [8-11]. The theory of functions of complex variables, a new branch of mathematics, ruled the mathematics of the nineteenth century, when mathematicians recognized it as the most prolific branch of mathematics, and called the century's mathematical enjoyment, and some people praised it is one of the most harmonious theory of abstract science. In the early 20th century, it has made great progress, one of Swedish mathematician, two of French mathematician and so on have made a lot of research work, opening up a broader field, and made important contributions to the development of this discipline.

2. Discussion

We will be including Cartesian coordinates and polar coordinates, complex number parameters and logarithms, differentiable functions, Cauchy-Riemann equations, etc.

We get in touch with the basic operations of complex numbers and the basic concepts with respect to cartesian and polar representations. We were first taught to distinguish the differences between real numbers and complex variables, and to familiarize complex variables with expressions in numbers such as $\sqrt{-1}$ and $\log -3$ ... After that, we mastered three methods expressing complex variables use methodology of number-shape combination. The real part is expressed as the x-axis of complex plane and the imaginary part is expressed by y. We can express all complex variables with the format of $z = x + iy$.

When expressing z1z2, $z1z2 = (x_1 + iy_1)(x_2 + iy_2)$

$$= (x_1x_2 - y_1y_2) + i(y_1x_2 + y_2x_1) \tag{1}$$

Similarly, $\frac{1}{z}$ can also be expressed with the format with $\mathrm{Re} + i \cdot \mathrm{Imag} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$

There is also an alternative way of representing a complex number: Assume there is a point z in the complex plane, $z = x + iy$, the modulus of z: $\rho = \sqrt{x^2 + y^2}$ is the distance from the point z to the origin of the plane, and the angle between the match and the x-axis is θ, then the trigonometric method of expressing complex variables comes out: $z = x + iy$

$$= \rho \cos\theta + i\rho \sin\theta = \rho(\cos\theta + i \sin\theta), \tag{2}$$

The third method of expressing complex numbers is to use the exponential form:

$$\cos\theta + i \sin\theta = e^{i\theta}, \tag{3}$$

Using this property, we can find out that $e^{i0} = 1, e^{i2\pi} = 1, e^{i\pi} = -1, e^{i\frac{\pi}{2}} = i$. Also, $z = \rho e^{i\theta}$, which can be called as the polar representation of z.

We get in touch with the complex conjugate of z, which is $\bar{z} = x - iy$, so we can conclude that:

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = \rho^2, \tag{4}$$

As we all know, the triangle inequality can help expressing the relationship between vector quantity as $|a + b| \leq |a| + |b|$. In this case,

$$|z_1| = |z_2 - (z_2 - z_1)| \leq |z_2| + |z_2 - z_1|, \tag{5}$$

Also, there is a small reminder on inequalities of complex variables: we cannot write $z_1 < z_2$, but can only compare their modulus.

We define $z^\alpha = \rho^\alpha e^{i(\theta+2\pi n)\alpha}$, when $\alpha \neq$rational numbers and n is an integer.

We define $\log z = \log(\rho e^{i(\theta+2\pi n)})$, when $z \neq 0$. So $\log -1 = \log e^{i(\pi+2\pi n)} = i\pi(2n + 1)$.

We define $e^z = e^{x+iy} = e^x(\cos y + i \sin y) = e^x \cos y + ie^x \sin y$ After we skilled in polar representations, we can start studying about functions related to complex variables.
The basic formula representing the functions with explanatory variable of z can be expressed as

$$W = f(z) = U(z) + iV(z) = U(x, y) + iV(x, y).$$

Example:

$$W = f(z) = z^2 = (x + iy)^2 = (x + iy)(x + iy) = (x^2 - y^2) + 2ixy, \tag{6}$$

So

$$U(x, y) = x^2 - y^2 = \rho^2((\cos\theta)^2 - (\sin\theta)^2) = \rho^2 \cos 2\theta. \tag{7}$$

And $V(x, y) = 2xy = 2\rho^2 \cos\theta \sin\theta = \rho^2 \sin 2\theta$. Similarly, when given $f(z) = z + \frac{1}{z}$, we can find out that $U(\rho, \theta) = \cos\theta(\rho + \rho^{-1})$ $V(\rho, \theta) = \sin\theta(\rho - \rho^{-1})$.To study the functions of complex variable visually, we used complex plane to perform the image of f(z) given conditions: if ρ is fixed and θ moves $0 \le \theta \le 2\pi$, what is the image by f(z) of this circle. Given $(\frac{U}{\rho+\rho^{-1}})^2 + (\frac{V}{\rho-\rho^{-1}})^2 = 1$, the graph will be ellipse.We have also defined the continuity and differentiability of the function $W = f(z)$.
f(z) is continuous at $z = z_0$ if $\lim_{z \to z_0} f(z) = f(z_0)$. Also, f(z) is differentiable at $z = z_0$ if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists.When $f(z) = U(z) + iV(z)$ is differentiable at $z = z_0$, then the Cauchy Riemann formula exists: $U_x(z_0) = V_y(z_0)$. $U_y(z_0) = -V_x(z_0)$.After leaning the Cauchy Riemann formula, we learned about the integration of functions along paths:
Assume that $\gamma'(t) = x'(t) + iy'(t)$ exists for all $t \in [a, b]$, and let f(z) be a complex valued function of z which is continuous, then $\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$.
The Cauchy Goursat formula is defined as:
If f(x) is holomorphic and differentiable at every point in U, then in a closed contour within U without holes:

$$\int_\gamma f(z)dz = 0, \tag{8}$$

Here is an example with holes within U:

$$f(z) = 1/z$$
$$U = \{z | 1 < |z| < 2\}, \tag{9}$$

Assume the hole can be expressed by γ_2, and the rest of the space is expressed by γ_1, then $\int_{\gamma_1} \frac{1}{z} dz = 0$, and $\int_{\odot \text{with } r=3/2} \frac{1}{z} dz = 2\pi i \neq 0$. Then, we are introduced with Cauchy's integral formula: when there is a circle $\gamma(t) = z_0 + re^{it}$, $(0 \le t \le 2\pi)$, and f(z) is holomorphic in an open region that includes the closed disc $|z - z_0| \le r$, then

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0} dz, \tag{10}$$

(Note that the integral is not holomorphic at all points inside γ)
After this, we are introduced with the Taylor's formula of Cauchy-Goursat theorem:
If f(z) is holomorphic in the region $|z - z_0| < R$, $\lim_{z \to z_1} \frac{f(z) - f(z_1)}{z - z_1}$ exists, then for $|z - z_0| < R$,

$$f(z) - \sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots \tag{11}$$

Also, the Laurent expansion and be expressed as:

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$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z-z_0)^n} + \ldots \quad (12)$$

In lecture 4, we are introduced with the Cauchy residual theorem:
When f(z) is differentiable at every point in U except at points when the function is not holomorphic, z_1, z_2 and z_p.
Then

$$\int_{\gamma} f(z) = 2\pi i(Res(f_1 z_1) + Res(f_1 z_2) + Res(f_1 z_p)). \quad (13)$$

In lecture 5, we learned how to expand a complex function into separate parts and combine them together to solve the problem—find the existing points in a certain area.

The Laurent theorem is represent complex functions on rings, most of its properties are arising from properties of power series. We can expand the complex function into single parts, and use capital letters to express the coefficient. By combining the single parts together, we can get a function of the coefficients. It should be pointed out that it's hard to solve the function with too many unknowns, so we need help from the matrix. By using the inverse matrix, we can produce the results of the coefficients.

The next step is to check whether the points are in the area the questions provide, if the points exist in the area, it's important the use 2pi to multiple the coefficients In lecture 5, we learned how to expand a complex function to separate part and combine them together to solve the problem—find the existing points in the certain area.

The complex numbers are the set $\{x + iy | x, y \in R\}$, where $i = \sqrt{-1}$. Re(z)=x, Im(z)=y.

$$|z| = \sqrt{x^2 + y^2} \quad (14)$$

For $z \in C, z = |z|e^{i\theta}$ is called the polar form of z. the angle $\theta \in R$is called the argument of z.
The theory of functions of complex variables arose in the eighteenth century. In 1774, Euler considered in one of his papers two equations derived from the integrals of functions of complex variables. Earlier than him, the French mathematician D'Alembert had already obtained them in his treatise on hydrodynamics. Therefore, these two equations were later referred to as the "D'Alembert-Euler equations".
C is complete: all Cauchy sequences converge
a set $A < C$ is closed if and only if for any convergent sequence $(z_n)_{n \in (x)} < A$ one has $\lim_{n \to \infty} z_n \in A$
For $A < C$, the following are equivalent:
A is compact
Every sequence $(z_n)_{n \in (x)} < A$ has a subsequence $(z_{n_k})_{k \in N}$ converging to a point A
Every open cover $\{U_i < C | i \in I\}$ of A has finite sub cover $\{U_{i1}, U_{i2}, \ldots, U_{in}\}$
If $A_1 > A_2 > \cdots > A_n > \cdots$ is sequence of non-empty compact sets in C with the property that $\lim_{n \to \infty} \mathrm{diam}(A_n) = 0$, then there is a unique point $\omega \in C$ that $\omega \in A_n$ for all $n \in N$. That is $\prod_{n \in N} A_n = \{\omega\}$.
The full development of the theory of functions of complex variables came in the nineteenth century, and just as the direct extension of calculus ruled the mathematics of the eighteenth century, the new branch of functions of complex variables ruled the mathematics of the nineteenth century. In the nineteenth century, these two equations were studied in more detail when Cauchy and Riemann studied hydrodynamics, so they were also called "Cauchy-Riemann conditions".
If f and g are holomorphic on A, then f+g is holomorphic on A with (f+g)'=f'+g'; f·g is holomorphic on A with (f·g)'=f'g+fg'.
f/g is holomorphic on A with (f/g)'=$\frac{f'g-fg'}{g^2}$
Cauchy-Riemann Equations

$$u(x,y) \coloneqq Re(f(x+iy))v(x,y) \coloneqq Im(f(x+iy)) \quad (15)$$

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So that $f(x+iy) = u(x,y) + iv(x,y)$. Suppose new that f is holomorphic at $z_0 = x_0 + iy_0$. For the moment, we will view $f(x,y) := f(x+iy)$ as a function on R^2 so that it means sense to consider its partial derivatives:$\frac{\partial f}{\partial x}(x0,y0) := \lim_{R\ni h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$, $\frac{\partial f}{\partial y}(x0,y0) := \lim_{R\ni h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$.
We can relate there to f'(z) by considering different paths $C\ni h \to 0$.

$$f'(z) = \lim_{R\ni h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \lim_{R\ni h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \frac{\partial f}{\partial x}(x0,y0) \quad (16)$$

Next by considering $iR \ni ih \to 0$ we have

$$f'(z) = \lim_{iR\ni ih\to 0} \frac{f(z+ih)-f(z)}{ih} = \frac{1}{i} \lim_{iR\ni ih\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \frac{1}{i}\frac{\partial f}{\partial y}(x0,y0) \quad (17)$$

Therefore we must have

$$\frac{\partial f}{\partial x}(x0,y0) = \frac{1}{i}\frac{\partial f}{\partial y}(x0,y0) \quad (18)$$

By substituting $f = u + iv$ and taking real and imaging parts, one obtains the Cauchy-Riemann equations $\frac{\partial u}{\partial x}(x0,y0) = \frac{\partial v}{\partial y}(x0,y0)$. And $\frac{\partial u}{\partial y}(x0,y0) = -\frac{\partial v}{\partial x}(x0,y0)$. Thus if f is holomorphic an open set $A < C$, we have shown its real and imaging parts have partial derivate satisfying the Cauchy-Riemann equations on all of A. Moreover, there partial derivatives will be continuous as consequence of holomorphic functions being infinitely differentiable.
At the beginning of the twentieth century, the theory of functions of complex variables has made great progress again, Weierstrass's students, the Swedish mathematician Leffler, the French mathematician Poincare, Hadamard and so on have done a lot of research work, opened up a broader field of study, for the development of the discipline has made a contribution.
Let $\Omega < \mathbb{R}^2$ be an open subset. Suppose f=u+iv is holomorphic on Ω with: $f' = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x}$. By describing methods for calculating an improper integral in an imaginary way. He introduced his Cauchy-Riemann equations by publishing his research paper Resume. He later incorporate new geometrical representation of the complex numbers in his later works.
Suppose $f(z) = \sum_{n=0} a_n(z-z_0)^n$, where $(a_n)_{n\in\propto} < \mathbb{C}$. We say the power series converges absolutely at z if $\sum_{n=0}^{\infty}|a_n||z-z_0|^n < \infty$. The Hadamard Formula states that the radius of convergence for a power series: $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Charles Briot and Jean-Claude Bouquet are two who introduced the concept of "holomorphic". Deriving from Greek vocabularies, holomorphic functions is defined as "whole" in a domain of complex plane. Holomorphic functions sometimes imply to "analytic function" in current year. Let $f(z) = \sum_{n=0}^{a} a_n(z-z_0)^n$ be a power series with radius of convergence R>0. Then f is holomorphic on $\mathbb{D}R(z_0)$ with $f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}$. Moreover, f' also has radius of convergence R.

3. Conclusion

We first briefly introduce the history of complex analysis by introducing when and how it origins. By finding the root of equation and observing the square root of negative numbers, complex numbers emerged. Then, we introduce the applications of complex numbers on physics and machineries invention. We list some of the applications of complex numbers on propagation of acoustic waves, Fre-bounded problems and so on. Also, the branches on mathematics complex numbers applied to are introduced. As for the history of complex numbers, they have the history of 150 years.
After introduction part and stasis, we start the review part of complex analysis. The review parts include Cartesian coordinates, polar coordinates, complex number parameters, Cauchy-Riemann equations, power series, application of Cauchy integral formula, etc.

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